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# HOMOMORPHISMS AND FUZZY HOMOMORPHISMS OF FUZZY LIE ALGEBRAS

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## Abstract

In this paper, we obtain some new results concerning homomorphisms and fuzzy homomorphisms of fuzzy Lie algebras. The concept of fuzzy homomorphism is introduced to Lie algebras and its related properties are discussed. Moreover, some properties of fuzzy Lie subalgebras and $(\alpha, \beta)$ fuzzy Lie subalgebras under homomorphisms of Lie algebras are studied.

**Key words:** Homomorphism, Fuzzy homomorphism, Lie algebras, Fuzzy subalgebras, $(\alpha, \beta)$ Fuzzy subalgebras.

**AMS Subject Classification:** 17B99.

### 0. Introduction

A Lie algebra over a field $\mathbf{F}$ is a vector space $L$, endowed with a bilinear multiplication: $[\cdot,\cdot] : L \times L \to g$, i.e.,for any $x, y, z \in L$, $(x, y) \to [x, y]$ satisfying the following properties:

$$[x, x] = 0 \quad (Anti - commutativity),$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad (Jacobi \quad identity).$$

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The theory of Lie algebras, which was initiated by the Norwegian mathematician Marius Sophus Lie, originally served as a tool in the analysis of the local structure of Lie groups. The objects in Lie theory are fundamental, interesting and innovating in both mathematics and physics. It has many applications to the spectroscopy of molecules, atoms, nuclei and hadrons, etc. One of the key concepts in the application of Lie algebraic methods in physics is that of spectrum generating algebras and their associated dynamic symmetries.

Zadeh [14] formulated the notion of fuzzy sets and after that many scholars developed fuzzy system of different algebraic structures. Recently, many mathematicians have been involved in extending the concepts and results of Lie algebras [1-3,5-7,9,11-13]. A purpose of the present paper is to obtain some new results concerning homomorphisms and fuzzy homomorphisms of fuzzy Lie algebras.

We proceed as follows. In Section 1, we recall some basic definitions and notations which will be used in what follows. In Section 2, we introduce the definition of fuzzy homomorphism of Lie algebras, discuss its related properties and obtain the basic theorem of fuzzy homomorphism. In Section 3, some properties of fuzzy Lie subalgebras are studied. In last section, we investigate the structure and properties of $(\alpha, \beta)$ fuzzy Lie subalgebras under homomorphisms of Lie algebras.

## 1 Preliminaries

In this section, some relevant definitions and notations are reproduced. Throughout, $g$ is a Lie algebra over a field $\mathbf{F}$ of positive characteristic zero, with commutator $g \times g \longrightarrow g$ denoted by $(x, y) \mapsto [x, y]$.

**Definition** 1.2[12] A fuzzy Lie algebra $A$ of $g$ is a fuzzy subset $A \in I^g$ which satisfies for all $x, y \in g$ and $\alpha, \beta \in \mathbf{F}$.

(i) $A(\alpha x + \beta y) \geq A(x) \wedge A(y)$

(ii) $A([x, y]) \geq A(x) \wedge A(y)$

(iii) $A(0) = 1$.

**Definition** 1.2[12] A fuzzy ideal $A$ of $g$ is a fuzzy subset $A \in I^g$ which satisfies for all $x, y \in g$ and $\alpha, \beta \in \mathbf{F}$.

(i) $A(\alpha x + \beta y) \geq A(x) \wedge A(y)$

(ii) $A([x,y]) \geq A(x) \vee A(y)$

(iii) $A(0) = 1$.

**Definition** 1.3[1] Let $X$ and $Y$ be any two sets and $f : X \longrightarrow Y$ be a surjective mapping. For $\forall\, x, y \in X$ and $A$ is a fuzzy subset of $X$, if $f(x) = f(y)$ holds, we have $A(x) = A(y)$. Then we call $A$ $f-incariant$.

Let $A$ be a fuzzy ideal of $g$, for each $x \in g$, define the fuzzy subset $x + A \in I^g$ by $(x + A)(y) = A(y - x)$. Let $A$ be an ideal in $g$, for all $x, y \in g$ and $\alpha \in \mathbf{F}$. Define:

(i) $(x + A) + (y + A) = (x + y) + A$,

(ii) $[x + A, y + A] = [x, y] + A$,

(iii) $\alpha(x + A) = \alpha x + A$.

The above three operations are well defined and make $g/A$ into a Lie algebra called the fuzzy-quotient Lie algebra of $g$ by $A$ [12].

**Definition** 1.4[9] Let $X$ and $Y$ be any two sets and $f : X \longrightarrow Y$ be a mapping. For any fuzzy subset $A$ of $X$, we define the fuzzy subset $B$ of $Y$ by

$$B(y) = \begin{cases} \sup\ \{\, \mathrm{A(x)} \mid \quad \mathrm{f(x)=y}\ \} & f^{-1}(y) \neq \emptyset \\ 0 & f^{-1}(y) = \emptyset,\ \ \forall y \in Y. \end{cases}$$

$B(y)$ is called the image of $A$ under $f$ and denoted by $f(A)$. For any fuzzy subset $C$ of $Y$, we define the fuzzy set $A$ of $X$ by $A(x) = C(f(x))$, for all $x \in X$, where $A(x)$ is called the preimage of $C$ under $f$ and denoted by $f^{-1}(C)$.

**Proposition** 1.1[1] Let $A$ be a fuzzy subspace of vector space $V$ over a field $\mathbf{F}$. Then

(1) $A(0) = \sup\{\ A(x) \mid x \in V\ \}$

(2) $A(\alpha x) = A(x)$, For all $\alpha \in \mathbf{F}\backslash\{0\}$ and $x \in V$

(3) Suppose $x, y \in V$ and $A(x) \neq A(y)$, then $A(x + y) = A(x) \wedge A(y)$.

## 2 Fuzzy Homomorphism of Lie Algebra

**Definition** 2.1[4] Assume that $\theta \in [0, 1)$ is determined. Let $g_1, g_2$ be two Lie algebras over a field $\mathbf{F}$ and $f$ be a fuzzy relation between $g_1$ and $g_2$. We will consider following conditions:

(i) For all $x \in g_1$, $\exists\, y \in g_2$, such that $f(x, y) > \theta$;

(ii) For all $y \in g_2$, $\exists\, x \in g_1$, such that $f(x,y) > \theta$;

(iii) For all $x \in g_1$ and $y_1,\ y_2 \in g_2$, from $f(x,y_1) > \theta,\ \ f(x,y_2) > \theta$, we have $y_1 = y_2$;

(iv) For all $x_1, x_2 \in g_1,\ y \in g_2$, from $f(x,y_1) > \theta,\ f(x,y_2) > \theta$, we have $x_1 = x_2$.

Let $f$ be a fuzzy relation between $g_1$ and $g_2$. When condition (i) and (iii) establish, we call $f$ $\theta$-fuzzy mapping from $g_1$ to $g_2$ and for short fuzzy mapping. Remark $f : g_1 \longrightarrow g_2$; When condition (i), (ii)and (iii) establish, then $f$ is called surjective fuzzy mapping; When condition (i), (iii) and (iv) establish, then $f$ is called bijective fuzzy mapping.

We can conclude that fuzzy mapping is a special mapping from above definition.

**Remark:** We will ask $y_x$ represents for all $x \in g_1$, there exists $y_x \in g_2$ such that $f(x,y_x) > \theta$.

**Definition 2.2** Let $g_1$ and $g_2$ be two lie algebras over a field $\mathbf{F}$ and $f$ be a fuzzy mapping from $g_1$ to $g_2$. For all $x_1,\ x_2 \in g_1,\ y \in g_2$ and $\alpha,\ \beta \in \mathbf{F}$, we have

(i) $f(\alpha x_1 + \beta x_2, y) = \sup\{f(x_1,y_1) \wedge f(x_2,y_2) \mid y = \alpha y_1 + \beta y_2\}$

(ii) $f([x_1,x_2],y) = \sup\{f(x_1,y_1) \wedge f(x_2,y_2) \mid y = [y_1,y_2]\}$

Then we call $f$ is a fuzzy homomorphism from $g_1$ to $g_2$.

If a fuzzy homomorphism $f : g_1 \longrightarrow g_2$ is a surjective fuzzy mapping, $f$ is called surjective fuzzy homomorphism; if $f$ is a bijective mapping, $f$ is called fuzzy isomorphism.

**Theorem 2.1** Let $f : g_1 \longrightarrow g_2$ be a lie fuzzy homomorphism. Then the following statements hold:

(1) $f(0_1,0_2) > \theta$;

(2) $f(x,y) > \theta \Rightarrow f(-x,-y) > \theta$;

(3) $-y_x = y_{-x}$;

(4) $y_{x_1+x_2} = y_{x_1} + y_{x_2},\ \ y_{[x_1,x_2]} = [y_{x_1}, y_{x_2}]$;

(5) $[y_{x_1},[y_{x_2},y_{x_3}]] + [y_{x_2},[y_{x_3},y_{x_1}]] + [y_{x_3},[y_{x_1},y_{x_2}]] = 0$.

*Proof.* (1) For $0_1 \in g_1$, there exists $y \in g_2$ such that $f(0_1,y) > \theta$. Then

$$f(0_1, y+y) = f(0_1+0_1, y+y) \geq f(0_1,y) > \theta.$$

Hence $y + y = y$ and we have that $y = 0_2$. That is $f(0_1,0_2) > \theta$. Therefore we obtain $f(0_1,0_2) = 1$ because of the choice of $\theta$.

(2) Let $f(x,y) > \theta$ and $z \in g_2$ such that $f(-x,z) > \theta$. Then

$$f(0_1, y+z) = f(x+(-x), y+z) \geq f(x,y) \wedge f(-x,z) > \theta.$$

It follows from (1), we have $y + z = 0_2$. That is $z = -y$. Therefore $f(-x,-y) > \theta$.

(3) From $f(x,y_x) > \theta$, we have $f(-x,-y_x) > \theta$. Also we know $f(-x,y_{-x}) > \theta$, therefore $-y_x = y_{-x}$.

(4) Combine $f(x_1,y_{x_1}) > \theta$ with $f(x_2,y_{x_2}) > \theta$, we obtain that

$$f(x_1+x_2, y_{x_1}+y_{x_2}) \geq f(x_1,y_1) \wedge f(x_2,y_2) > \theta.$$

Therefore $y_{x_1+x_2} = y_{x_1} + y_{x_2}$.

By similar consideration, from $f([x_1,x_2],[y_{x_1},y_{x_2}]) \geq f(x_1,y_{x_1}) \wedge f(x_2,y_{x_2}) > \theta$ we obtain $y_{[x_1,x_2]} = [y_{x_1}, y_{x_2}]$.

(5) The result follows by means of (4). □

**Theorem 2.2** Let $f : g_1 \longrightarrow g_2$ be a Lie fuzzy homomorphism and $A$ be a fuzzy subalgebra of $g_1$. Then

(1) $f(A)$ be a fuzzy subalgebra of $g_2$.

(2) If $A$ is a fuzzy ideal of $g_1$ and $f$ is a surjective homomorphism, then $f(A)$ is a fuzzy ideal of $g_2$, where

$$f(A)(y) = \sup\{\, A(x) \mid f(x,y) > \theta\} \ \textit{for all}\ \ y \in g_2, \exists\, x \in g_1.$$

*Proof.* For all $y_1, y_2 \in g_2$ and $\alpha, \beta \in \mathbf{F}$, we have that

$$\begin{aligned} f(A)(\alpha y_1 + \beta y_2) &= \sup\{A(\alpha x_1 + \beta x_2) \mid f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) > \theta\} \\ &\geq \sup\{A(x_1) \wedge A(x_2) \mid f(x_1,y_1) > \theta, f(x_2,y_2) > \theta\} \\ &= \sup\{A(x_1) \mid f(x_1,y_1) > \theta\} \wedge \sup\{A(x_2) \mid f(x_2,y_2) > \theta\} \\ &= f(A)(y_1) \wedge f(A)(y_2). \end{aligned}$$

$$\begin{aligned} f(A)([y_1,y_2]) &= \sup\{A([x_1,x_2]) \mid f([x_1,x_2],[y_1,y_2]) > \theta\} \\ &\geq \sup\{A(x_1) \wedge A(x_2) \mid f(x_1,y_1) > \theta, f(x_2,y_2) > \theta\} \\ &= \sup\{A(x_1) \mid f(x_1,y_1) > \theta\} \wedge \sup\{A(x_2) \mid f(x_2,y_2) > \theta\} \\ &= f(A)(y_1) \wedge f(A)(y_2). \end{aligned}$$

$$\begin{aligned} f(A)(0) &= \sup\{A(x_1+x_2) \mid f(x_1+x_2, 0_2) > \theta\} \\ &= A(0) = 1. \end{aligned}$$

Hence $f(A)$ is a fuzzy subalgebra of $g_2$.

If $A$ is a fuzzy ideal of $g_1$ and $f$ is a surjective fuzzy homomorphism, for all $y_1$, $y_2 \in g_1$, there exist $x_1, x_2 \in g_1$ such that $f(x_1, y_1) > \theta$, $f(x_2, y_2) > \theta$. Since

$$\begin{aligned} f(A)([y_1, y_2]) &= \sup\{A([x_1, x_2]) \mid f([x_1, x_2], [y_1, y_2]) > \theta\} \\ &\geq \sup\{A(x_1) \vee A(x_2) \mid f(x_1, y_1) > \theta, f(x_2, y_2) > \theta\} \\ &= \sup\{A(x_1) \mid f(x_1, y_1) > \theta\} \vee \sup\{A(x_2) \mid f(x_2, y_2) > \theta\} \\ &= f(A)(y_1) \vee f(A)(y_2), \end{aligned}$$

we obtain that $f(A)$ is a fuzzy ideal of $g_2$. □

**Corollary 2.3** Let $f : g_1 \to g_2$ be a lie fuzzy homomorphism and $B$ be a fuzzy subalgebra (ideal) of $g_2$. Then $f^{-1}(B)$ is a fuzzy subalgebra (ideal) of $g_1$, where $f^{-1}(B)(x) = B(y_x)$.

**Theorem 2.4** Let $A$ be a fuzzy ideal of $g$. Then $g$ and $g/A$ are fuzzy homomorphic.

*Proof.* For all $x, y \in g$, define $f(x, y + A) := A(x - y)$, then we have

$$f(x, x + A) = A(x - x) = A(0).$$

If $y_1 + A = y_2 + A$, then

$$f(x, y_1 + A) = A(x - y_1) = y_1 + A(x) = y_2 + A(x) = A(x - y_2) = f(x, y_2 + A).$$

And if $f(x, y_1 + A) = f(x, y_2 + A) = A(0)$, then $A(x - y_1) = A(x - y_2) = A(0)$. Moreover $A(y_1 - y_2) = A((x - y_2) - (x - y_1)) \geq A(x - y_2) \wedge A(x - y_1) = A(0)$. That is $A(y_1 - y_2) = A(0)$. So

$$y_1 + A = y_2 + A.$$

We have that $f$ is a fuzzy mapping and obviously it is surjective.

For all $x_1, x_2 \in g$ and $\alpha, \beta \in \mathbf{F}$, if there exists$y_1, y_2 \in g$ such that

$y + A = (\alpha y_1 + A) + (\beta y_2 + A)$, then

$$\begin{aligned} f(\alpha x_1 + \beta x_2, y + A) &= f(\alpha x_1 + \beta x_2, (\alpha y_1 + \beta y_2) + A) \\ &= A(\alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2) \\ &\geq A(\alpha x_1 - \alpha y_1) \wedge A(\beta x_2 - \beta y_2) \\ &= A(x_1 - y_1) \wedge A(x_2 - y_2) \\ &= f(x_1, y_1 + A) \wedge f(x_2, y_2 + A). \end{aligned}$$

So we obtain that

$$f(\alpha x_1 + \beta x_2, y + A) \geq \sup \{f(x_1, y_1 + A) \wedge f(x_2, y_2 + A) \mid y + A = (\alpha y_1 + A) + (\beta y_2 + A)\}.$$

Moreover,

$$\begin{aligned} \sup \{f(x_1, y_1 + A) \quad \wedge \quad & f(x_2, y_2 + A) \mid y + A = (\alpha y_1 + A) + (\beta y_2 + A)\} \\ &\geq f(x_1, x_1 + A) \wedge f(x_2, \beta^{-1}\alpha(y_1 - x_1) + y_2 + A) \\ &= A(0) \wedge A(x_2 - \beta^{-1}\alpha(y_1 - x_1) - y_2) \\ &= A(\beta x_2 + \alpha x_1 - \alpha y_1 - \beta y_2) \\ &= f(\alpha x_1 + \beta x_2, y + A). \end{aligned}$$

So we know that

$$f(\alpha x_1 + \beta x_2, y + A) = \sup \{f(x_1, y_1 + A) \wedge f(x_2, y_2 + A) \mid y + A = (\alpha y_1 + A) + (\beta y_2 + A)\}.$$

In the same way, we have

$$f([x_1, x_2], y + A) = \sup \{f(x_1, y_1 + A) \wedge f(x_2, y_2 + A) \mid y + A = [y_1, y_2] + A = [y_1 + A, y_2 + A]\}.$$

Hence $f$ is a fuzzy homomorphism.

Therefore $g$ and $g/A$ are fuzzy homomorphic. □

**Theorem 2.5** ( The Basic Theorem of Fuzzy Homomorphism)

Let $f : g_1 \to g_2$ be a Lie fuzzy homomorphism, then

(1) $A$ is a fuzzy ideal of $g_1$. Where $A(x) = f(x, 0_2) \wedge f(-x, 0_2)$, for all $x \in g_1$.

(2) If $f$ is a surjective fuzzy homomorphism, then $g_1/A$ and $g_2$ are fuzzy isomorphic.

*Proof.* (1) For all $x_1$, $x_2 \in g_1$ and $\alpha$, $\beta \in \mathbf{F}$, we have that

$$\begin{aligned} A(\alpha x_1 + \beta x_2) &= f(\alpha x_1 + \beta x_2, 0_2) \wedge f(-\alpha x_1 - \beta x_2, 0_2) \\ &\geq f(x_1, 0_2) \wedge f(x_2, 0_2) \wedge f(-x_1, 0_2) \wedge f(-x_2, 0_2) \\ &= A(x_1) \wedge A(x_2). \end{aligned}$$

$$\begin{aligned} A([x_1, x_2]) &= f([x_1, x_2], 0_2) \wedge f(-[x_1, x_2], 0_2) \\ &= f([x_1, x_2], y_{x_1} 0_2) \wedge f([x_1, -x_2], y_{x_1} 0_2) \\ &\geq f(x_1, y_{x_1}) \wedge f(x_2, 0_2) \wedge f(x_1, y_{x_1}) \wedge f(-x_2, 0_2) \\ &= f(x_2, 0_2) \wedge f(-x_2, 0_2) = A(x_2). \end{aligned}$$

By similar consideration, we obtain $A([x_1, x_2]) \geq A(x_1)$. That is $A([x_1, x_2]) \geq A(x_1) \vee A(x_2)$. Moreover, $A(0_1) = f(0_1, 0_2) \wedge f(0_1, 0_2) = 1$. Hence $A$ is a fuzzy ideal of $g_1$.

(2) For all $x \in g_1$ and $y \in g_2$, define $h(x + A, y) := f(x, y)$. If there exist $x_1, x_2 \in g_1$ such that $x_1 + A = x_2 + A$, then $A(x_1 - x_2) = A(0) = f(0_1, 0_2)$. So

$$f(x_1 - x_2, 0_2) = f(x_2 - x_1, 0_2) = A(0) = f(0_1, 0_2).$$

Moreover, $f(x_1, y) \geq f(x_2, y) \wedge f(x_1 - x_2, 0_2) = f(x_2, y)$ and
$f(x_2, y) \geq f(x_1, y) \wedge f(x_2 - x_1, 0_2) = f(x_1, y)$. Hence $h$ is a fuzzy subalgebra of $g_1/A \times g_2$.

For all $x \in g_1$, we have $h(x + A, y_x) = f(x, y_x) = A(0)$. If $h(x + A, y_1) = h(x + A, y_2) = A(0), y_1, y_2 \in g_2$, then $f(x, y_1) = f(x, y_2) = A(0)$. So $y_1 = y_2$.
That is, $h$ is a fuzzy mapping and obviously it is surjective. If $f(x_1 + A, y) = f(x_2 + A, y) = A(0)$, $x_1, x_2 \in g_1$, then $f(x_1, y) = f(x_2, y) = A(0)$. Thus

$$\begin{aligned} f(x_1 - x_2, 0_2) &\geq f(x_1, y) \wedge f(-x_2, -y) \\ &= f(x_1, y) \wedge f(x_2, y) = A(0) \end{aligned}$$

By similar consideration, we obtain $f(x_2 - x_1, 0_2) = A(0)$,
i.e., $A(x_1 - x_2) = f(x_1 - x_2, 0_2) \wedge f(x_2 - x_1, 0_2) = A(0)$.
So $x_1 + A = x_2 + A$.

Therefore, $h$ is a fuzzy mapping. Also

$$\begin{aligned} h((\alpha x_1 + A) + (\beta x_2 + A), y) &= h(\alpha x_1 + \beta x_2 + A, y) = f(\alpha x_1 + \beta x_2, y) \\ &= \sup \{ f(x_1, y_1) \wedge f(x_2, y_2) \mid y = \alpha y_1 + \beta y_2 \} \\ &= \sup \{ h(x_1 + A, y_1) \wedge h(x_2 + A, y_2) \mid y = \alpha y_1 + \beta y_2 \}. \end{aligned}$$

By similar consideration, we obtain

$$\begin{aligned} h([x_1+A, x_2+A], y) &= h([x_1, x_2]+A, y) = f([x_1, x_2], y) \\ &= \sup\{f(x_1, y_1) \wedge f(x_2, y_2) \mid y = [y_1, y_2]\} \\ &= \sup\{h(x_1+A, y_1) \wedge h(x_2+A, y_2) \mid y = [y_1, y_2]\}. \end{aligned}$$

So $h$ is a fuzzy isomorphism and $g_1/A$ and$g_2$ are fuzzy isomorphic. □

## 3 Homomorphism of Fuzzy Lie Subalgebras

**Definition 3.1** Let $A$ and $B$ be a fuzzy ideal and a fuzzy subset of $g$ respectively. Then $B/A$ is a fuzzy subset of $g/A$ which is denoted by

$$(B/A)(x+A) = \sup\{B(y) \mid y+A = x+A\}, \forall x \in g.$$

**Theorem 3.1** Let $A$ and $B$ be fuzzy ideal and fuzzy subalgebra of $g$. Then $B/A$ is a fuzzy subalgebra of $g/A$.

*Proof.* For all $x, y \in g$ and $\alpha, \beta \in \mathbf{F}$, we obtain that

$$\begin{aligned} (B/A)(\alpha(x+A) &+ \beta(y+A)) = (B/A)(\alpha x + \beta y + A) \\ &= \sup\{B(\alpha x_1 + \beta y_1) \mid \alpha x_1 + \beta y_1 + A = \alpha x + \beta y + A\} \\ &\geq \sup\{B(x_1) \wedge B(y_1) | \alpha(x_1+A) + \beta(y_1+A) = \alpha(x+A) + \beta(y+A)\} \\ &= \sup\{B(x_1) \wedge B(y_1) \mid x_1+A = x+A, y_1+A = y+A\} \\ &= \sup\{B(x_1) \mid x_1+A = x+A\} \wedge \sup\{B(y_1) \mid y_1+A = y+A\} \\ &= (B/A)(x+A) \wedge (B/A)(y+A). \end{aligned}$$

$$\begin{aligned} (B/A)([x+A, y+A]) &= (B/A)([x, y]+A) \\ &= \sup\{B([x_1, y_1]) \mid [x_1, y_1]+A = [x, y]+A\} \\ &\geq \sup\{B(x_1) \wedge B(y_1) | x_1+A = x+A, y_1+A = y+A \\ &= \sup\{B(x_1) \mid x_1+A = x+A\} \wedge \sup\{B(y_1) \mid y_1+A = y+A\} \\ &= (B/A)(x+A) \wedge (B/A)(y+A). \end{aligned}$$

$$(B/A)(0+A) = \sup\{B(y) \mid y+A = 0+A\} = 1.$$

So $B/A$ is the fuzzy subalgebra of $g/A$. □

**Corollary 3.2** Let $A$ and $B$ be two fuzzy ideals of $g$. Then $B/A$ is a fuzzy ideal of $g/A$.

**Theorem 3.3** Let $A$ and $B$ be a fuzzy subalgebra and a fuzzy ideal of $g$ respectively. Then $A \approx A/B$. ($\approx$ means homomorphism.)

*Proof.* Assume that $f : g \longrightarrow A/B$, $x \mapsto x + B$ is a nature homomorphism. Obviously $f$ is a surjective homomorphism. For all $x \in g$, we have that

$$\begin{aligned} f(A)(x+B) &= \sup\{A(t) \mid f(t) = x+B\} \\ &= \sup\{A(t) \mid t+B = x+B\} = (A/B)(x+B)\}. \end{aligned}$$

So $f(A) = A/B$. Thus we have $A \approx A/B$. □

Let $A$ be a fuzzy subalgebra of $g$. $A_* = \{\, x \in g \mid A(x) = A(0) \,\}$.

**Proposition 3.4**[6] Let $A$ and $B$ be two fuzzy subalgebra of $g$. Then we have $(A \cap B)_* = A_* \cap B_*$.

**Proposition 3.5**[6] Let $A$ be a fuzzy ideal of $g$. Then $A_*$ is a ideal of $g$.

**Theorem 3.6** Let $A$ be a fuzzy ideal of $g$. Then $g/A_* \cong g/A$.

*Proof.* Clearly. □

**Theorem 3.7** Let$f : g_1 \longrightarrow g_2$ be a surjective Lie homomorphism and $A$ be a $f-invariant$ fuzzy ideal of $g_1$. Then we have $g_1/A \cong g_2/f(A)$.

*Proof.* Since $A$ is $f - invariant$, we have $A = f^{-1}(f(A))$. So

$$\begin{aligned} \mathrm{Ker}g &= \{x \in g_1 \mid g(x) = f(A)\} \\ &= \{x \in g_1 \mid f(x) + f(A) = f(A)\} \\ &= \{x \in g_1 \mid f(A)(f(x)) = f(A)(0_2)\} \\ &= \{x \in g_1 \mid f^{-1}(f(A))(x) = A(0_1)\} \\ &= \{x \in g_1 \mid A(x) = A(0_1)\} = A_*. \end{aligned}$$

Hence we have $g_1/A_* \cong g_2/f(A)$.

The result follows by means of Theorem 3.6. □

**Theorem 3.8** Let $f : g_1 \longrightarrow g_2$ be a surjective Lie homomorphism. Let A and B be a fuzzy subalgebra and a fuzzy ideal of $g$ respectively. If $B$ is $f - invariant$, then we have $A/B \cong f(A)/f(B)$.

*Proof.* From Theorem 3.7, we have that $g : g_1/B \longrightarrow g_2/f(B)$, $x + B \mapsto f(x) + f(B)$, for all $x \in g_1$ be a isomorphism. For all $y \in g_2$, there exists $x \in g_1$ to make $f(x) = y$. Therefore

$$\begin{aligned}
(f(A)/f(B))(y + f(B)) &= (f(A)/f(B))(f(x) + f(B)) \\
&= \sup\{f(A)(v) \mid v + f(B) = f(x) + f(B)\} \\
&= \sup\{\sup\{A(u) \mid f(u) = v\} \mid v + f(B) = f(x) + f(B)\} \\
&= \sup\{A(u) \mid f(u) + f(B) = f(x) + f(B)\} \\
&= \sup\{A(u) \mid f(B)(f(x) - f(u)) = f(B)(0_2)\} \\
&= \sup\{A(u) \mid B(x - u) = B(0_1)\} \\
&= \sup\{A(u) \mid u + B = x + B\} \\
&= (A/B)(x + B) \\
&= g^{-1}(g(A/B))(x + B) \\
&= g(A/B)(g(x + B)) = g(A/B)(f(x) + f(B)) \\
&= g(A/B)(y + f(B)).
\end{aligned}$$

That is $g(A/B) = f(A)/f(B)$.

So we have $A/B \cong f(A)/f(B)$. □

**Corollary 3.9** Let $f : g_1 \longrightarrow g_2$ be a surjective homomorphism. Let $A$ and $B$ be a fuzzy subalgebra and a fuzzy ideal of $g_2$ respectively. Then we have $f^{-1}(A)/f^{-1}(B) \cong A/B$.

*Proof.* Since $f$ is surjective, $f(f^{-1}(A)) = A$ *and* $f(f^{-1}(B)) = B$ establish.

If $f(x_1) = f(x_2)$, then $f^{-1}(B)(x_1) = B(f(x_1)) = B(f(x_2)) = f^{-1}(B)(x_2)$.

So we have $f^{-1}(B)$ is $f - invariant$.

From Theorem 3.8, we obtain $f^{-1}(A)/f^{-1}(B) \cong A/B$. □

**Theorem 3.10** Let $f : g_1 \longrightarrow g_2$ be a Lie homomorphism. Let $A$ and $B$ be fuzzy ideals of $g_1$ and $g_2$ respectively. Also $f(A) \subseteq B$. Then there exists a homomorphism $f^* : g_1/A \longrightarrow g_2/B$ such that $f^* \circ h_1 = h_2 \circ f$.

Let $h_1 : g_1 \longrightarrow g_1/A$ and $h_2 : g_2 \longrightarrow g_2/B$ be two nature homomorphisms.

*Proof.* For all $x \in g_1$, let $f^* : g_1/A \to g_2/B, \quad x + A \mapsto f(x) + B$.

If $x_1, x_2 \in g$ and $x_1 + A = x_2 + A$, then $A(x_1 - x_2) = A(0_1)$. Hence

$$\begin{aligned} B(f(x_1) - f(x_2)) &= B(f(x_1 - x_2)) \\ &\geq f(A)(f(x_1 - x_2)) \\ &= f^{-1}(f(A))(x_1 - x_2) \\ &\geq A(x_1 - x_2) = A(0_1) = B(0_2). \end{aligned}$$

In another way, we have $B(f(x_1) - f(x_2)) = B(0_2)$, thus $f(x_1) + B = f(x_2) + B$.
We obtain that $f^*$ is a mapping from $g_1/A$ to $g_2/B$.

For all $x_1, x_2 \in g_1$ and $\alpha, \beta \in \mathbf{F}$, we have that

$$\begin{aligned} f^*(\alpha(x_1 + A) + \beta(x_2 + A)) &= f^*(\alpha x_1 + \beta x_2 + A) \\ &= f(\alpha x_1 + \beta x_2) + B \\ &= (\alpha f(x_1) + \beta f(x_2)) + B \\ &= \alpha(f(x_1) + B) + \beta(f(x_2) + B) \\ &= \alpha f^*(x_1 + A) + \beta f^*(x_2 + A). \end{aligned}$$

$$\begin{aligned} f^*([x_1 + A, x_2 + A]) &= f^*([x_1, x_2] + A) \\ &= f([x_1, x_2]) + B \\ &= ([f(x_1), f(x_2)]) + B \\ &= [f(x_1) + B, f(x_2) + B] \\ &= [f^*(x_1 + A), f^*(x_2 + A)]. \end{aligned}$$

So $f^*$ is a Lie homomorphism.

For all $x \in g_1$, we have $(f^* \circ h_1)(x) = f^*(h_1)(x) = f^*(x + A) = f(x) + B$ and
$(h_2 \circ f)(x) = h_2(f(x)) = f(x) + B$.
Hence $f^* \circ h_1 = h_2 \circ f$. □

**Lemma 3.11** Let $A$ and $B$ be two fuzzy ideals of g and $A \supseteq B$. Then for all $x, y \in g$, we have $x + B + A/B = y + B + A/B \Longleftrightarrow x + A = y + A$.

*Proof.* If $x + B + A/B = y + B + A/B$, then $(A/B)(x - y + B) = (A/B)(B) = A(0)$.
So we have $\sup\{A(t) \mid t + B = x - y + B\} = A(0)$.

If $t + B = x - y + B$, then $B(x - y - t) = B(0)$. Hence

$$\begin{aligned} A(x-y) \geq A(x-y-t) \wedge A(t) &\geq B(x-y-t) \wedge A(t) \\ &= B(0) \wedge A(t) = A(t). \end{aligned}$$

That is $A(x - y) \geq \sup\{A(t) \mid t + B = x - y + B\} = A(0)$, therefore $x + A = y + A$.

Contrarily, if $x + A = y + A$, then $A(x - y) = A(0) = (A/B)(B)$.

Thus $(A/B)(x - y + B) = \sup\{A(t) \mid t + B = x - y + B\} = A(0) = A(x - y) = (A/B)(B)$.

That is $(A/B)((x + B) - (y + B)) = (A/B)(B)$.

Therefore $x + B + A/B = y + B + A/B$. □

**Theorem 3.12** Let $A$ and $B$ be two fuzzy ideals of $g$ and $A \supseteq B$. If $C$ is a fuzzy subalgebra of $g$, then we have $(C/B)/(A/B) \cong C/A$.

*Proof.* Let $f : (g/B)/(A/B) \to g/A,\ r + B + A/B \mapsto r + A,$ for all $r \in g$.

From Lemma 3.11, we obtain that $f$ is a injective mapping and obviously it is surjective.

For all $x, y \in g$ and $\alpha, \beta \in \mathbf{F}$, we have that

$$\begin{aligned} f(\alpha(x + B + A/B) + \beta(y + B + A/B)) &= f(\alpha x + \beta y + B + A/B) \\ &= \alpha x + \beta y + A \\ &= \alpha(x + A) + \beta(y + A) \\ &= \alpha f(x + B + A/B) + \beta f(y + B + A/B). \end{aligned}$$

$$\begin{aligned} f([x + B + A/B, y + B + A/B]) &= f([x, y] + B + A/B) \\ &= [x, y] + A = [x + A, y + A] \\ &= [f(x + B + A/B), f(y + B + A/B)]. \end{aligned}$$

So $f$ is a isomorphism. Moreover,

$$\begin{aligned} ((C/B)/(A/B))(x + B + A/B) &= \sup\{(C/B)(r + B) \mid r + B + A/B = x + B + A/B\} \\ &= \sup\{\sup\{C(u) \mid u + B = r + B\} \mid r + B + A/B = x + B + A/B\} \\ &= \sup\{C(u) \mid u + B + A/B = x + B + A/B\} \\ &= \sup\{C(u) \mid u + A = x + A\} \\ &= (C/A)(x + A) = (C/A)(f(x + B + A/B)) \\ &= f^{-1}(C/A)(x + B + A/B). \end{aligned}$$

That is $(C/B)/(A/B) = f^{-1}(C/A)$. So $f((C/B)/(A/B)) = (C/A)$.

Thus $(C/B)/(A/B) \cong C/A$. □

## 4 Homomorphism of $(\alpha, \beta)$ Fuzzy Lie Subalgebra

**Definition** 4.1[3] Let $\alpha, \beta \in [0,1]$ and $\alpha < \beta$. Let $A$ be a fuzzy subset of a Lie algebra $g$. Then $A$ is called a fuzzy Lie subalgebra with thresholds of $g$ for short $(\alpha, \beta)$ fuzzy subalgebra, if for all $x, y \in g$ and $\gamma \in \mathbf{F}$,

(1a) $A(x+y) \vee \alpha \geq A(x) \wedge A(y) \wedge \beta$,

(1b) $A(\gamma x) \geq A(x) \wedge \beta$,

(2) $A([x,y]) \vee \alpha \geq A(x) \wedge A(y) \wedge \beta$.

Moreover, $A$ is a fuzzy ideal with thresholds of $g$, if $A$ satisfies the conditions (1a), (1b) and satisfies the following condition:

(3) $\mu([x,y]) \vee \alpha \geq (\mu(x) \vee \mu(y)) \wedge \beta$ for all $x, y \in g, \gamma \in \mathbf{F}$.

**Theorem 4.1** Let $f : g_1 \longrightarrow g_2$ be a lie homomorphism and $A$ be a $(\alpha, \beta)$ fuzzy subalgebra of $g_1$. Then $f(A)$ is a $(\alpha, \beta)$ fuzzy subalgebra of $g_2$. Specially, if $f$ is a surjective Lie homomorphism and $A$ is a $(\alpha, \beta)$ fuzzy ideal of $g_1$, then $f(A)$ is a $(\alpha, \beta)$ fuzzy ideal of $g_2$.

*Proof.* For all $x, y \in g_2, \alpha, \beta \in [0,1]$, we have that

$$\begin{aligned} f(A)(x+y) \vee \alpha &= \sup\{A(x_1+y_1) | f(x_1+y_1) = x+y\} \vee \alpha \\ &= \sup\{A(x_1+y_1) \vee \alpha | f(x_1+y_1) = x+y\} \\ &\geq \sup\{(A(x_1) \wedge A(y_1)) \wedge \beta | f(x_1) = x, f(y_1) = y\} \\ &= \sup\{A(x_1) | f(x_1) = x\} \wedge \sup\{A(y_1) | f(y_1) = y\} \wedge \beta \\ &= f(A)(x) \wedge f(A)(y) \wedge \beta. \end{aligned}$$

$$\begin{aligned} f(A)(\gamma x) \vee \alpha &= \sup\{A(\gamma x_1) | f(\gamma x_1) = \gamma x\} \vee \alpha \\ &= \sup\{A(\gamma x_1) \vee \alpha | f(\gamma x_1) = \gamma x\} \\ &\geq \sup\{A(x_1) \wedge \beta | f(x_1) = x\} \\ &= \sup\{A(x_1) | f(x_1) = x\} \wedge \beta \\ &= f(A)(x) \wedge \beta. \end{aligned}$$

By similar consideration, we obtain $f(A)([x,y]) \vee \alpha \geq (f(A)(x) \wedge f(A)(y)) \wedge \beta$.

Hence $f(A)$ is a $(\alpha, \beta)$ fuzzy subalgebra of $g_2$.

If $f$ is a surjective Lie homomorphism and $A$ is a $(\alpha, \beta)$ fuzzy ideal of $g_1$, for all $x, y \in g_2$, and $f^{-1}(x) \neq \emptyset, f^{-1}(y) \neq \emptyset$, then we obtain that

$$\begin{aligned} f(A)([x,y]) \vee \alpha &= \sup\{A([x_1,y_1]) \mid f([x_1,y_1]) = [x,y]\} \vee \alpha \\ &= \sup\{A([x_1,y_1]) \vee \alpha \mid f([x_1,y_1]) = [x,y]\} \\ &\geq \sup\{A(x_1) \mid f(x_1) = x\} \vee \sup\{A(y_1) \mid f(y_1) = y\} \wedge \beta \\ &= f(A)(x) \vee f(A)(y) \wedge \beta. \end{aligned}$$

So $f(A)$ is a $(\alpha, \beta)$ fuzzy ideal of $g_2$. □

**Theorem 4.2** Let $f : g_1 \longrightarrow g_2$ be a lie homomorphism and $B$ is a $(\alpha, \beta)$ fuzzy subalgebra (ideal). Then $f^{-1}(B)$ is a $(\alpha, \beta)$ fuzzy subalgebra (ideal) of $g_1$.

*Proof.* We only prove the situation of $(\alpha, \beta)$ fuzzy subalgebra.

For all $x, y \in g_1$, we have that

$$\begin{aligned} f^{-1}(B)(x+y) \vee \alpha &= B(f(x+y)) \vee \alpha = B(f(x) + f(y)) \vee \alpha \\ &\geq B(f(x) \wedge B(f(y)) \wedge \beta \\ &= (f^{-1}(B))(x) \wedge (f^{-1}(B))(y) \wedge . \end{aligned}$$

$$\begin{aligned} f^{-1}(B)(\gamma x) \vee \alpha &= B(f(\gamma x)) \vee \alpha \\ &= B(\gamma f(x)) \vee \alpha \geq B(f(x)) \wedge \beta \\ &= (f^{-1}(B))(x) \wedge \beta. \end{aligned}$$

$$\begin{aligned} f^{-1}(B)([x,y]) \vee \alpha &= B(f([x,y])) \vee \alpha \\ &= B([f(x), f(y)]) \vee \alpha \geq B(f(x)) \wedge B(f(y)) \wedge \beta \\ &= (f^{-1}(B))(x) \wedge (f^{-1}(B))(y) \wedge \beta. \end{aligned}$$

Thus $f^{-1}(B)$ is a $(\alpha, \beta)$ fuzzy subalgebra of $g_1$. □

Let $A$ be a $(\alpha, \beta)$ fuzzy subalgebra of $g$. For all $x \in g$, define fuzzy subset $x \oplus A$ of g as follows:

$$(x \oplus A)(y) = (A(y - x) \vee \alpha) \wedge \beta, \quad \textit{for all } y \in g.$$

**Theorem 4.3** Let $A$ is a $(\alpha, \beta)$ fuzzy subalgebra of $g$. For all $x, y \in g$, then $A(A(x - y) \vee \alpha) \wedge \beta = (A(0) \vee \alpha) \wedge \beta \Longleftrightarrow (A(y - x) \vee \alpha) \wedge \beta = (A(0) \vee \alpha) \wedge \beta)$.

*Proof.* Clearly. □

**Theorem 4.4** Let $A$ is a $(\alpha, \beta)$ fuzzy subalgebra of $g$. For all $x, y \in g$, then

$$x \oplus A = y \oplus A \Longleftrightarrow (A(x-y) \vee \alpha) \wedge \beta = (A(0) \vee \alpha) \wedge \beta.$$

*Proof.* Let $(A(x-y) \vee \alpha) \wedge \beta = A(A(0) \vee \alpha) \wedge \beta$, then for all $a \in g$,

$$\begin{aligned}
(y \oplus A)(a) &= (A(a-y) \vee \alpha) \wedge \beta \\
&= (A((a-x)+(x-y)) \vee \alpha) \wedge \beta \\
&\geq (A((a-x)+(x-y)) \vee \alpha) \wedge \beta \\
&= ((A(a-x) \vee \alpha) \wedge \beta) \wedge ((A(x-y) \vee \alpha) \wedge \beta) \\
&= ((A(a-x) \vee \alpha) \wedge \beta) \wedge ((A(0) \vee \alpha) \wedge \beta) \\
&= (A(a-x) \vee \alpha) \wedge \beta = (x \oplus A)(a).
\end{aligned}$$

So $y \oplus A \supseteq x \oplus A$.

From the above conclusion and theorem 4.3, we see $x \oplus A \supseteq y \oplus A$.

Hence we have $x \oplus A = y \oplus A$.

Conversely, if $x \oplus A = y \oplus A$, then we have

$(A(x-y) \vee \alpha) \wedge \beta = (y \oplus A)(x) = (x \oplus A)(x) = (A(0) \vee \alpha) \wedge \beta.$ □

**Theorem 4.5** Let $A$ be a $(\alpha, \beta)$ fuzzy ideal of $g$, for all $x, y, a, b \in g$, then the following statements are equalment:

(1)

$$\begin{cases} x \oplus A = a \oplus A \\ y \oplus A = b \oplus A \end{cases}$$

(2)

$$\begin{cases} (x+y) \oplus A = (a+b) \oplus A \\ [x, y] \oplus A = [a, b] \oplus A. \end{cases}$$

*Proof.* Since $x \oplus A = a \oplus A$ *and* $y \oplus A = b \oplus A$,

so $(A(x-a) \vee \alpha) \wedge \beta = (A(0) \vee \alpha) \wedge \beta$ and

$(A(y-b) \vee \alpha) \wedge \beta = (A(0) \vee \alpha) \wedge \beta$. Then

$$\begin{aligned}((A(x+y)-(a+b)) \vee \alpha) \wedge \beta &= ((A(x-a)+(y-b)) \vee \alpha) \wedge \beta \\ &\geq ((A(x-a) \wedge A(y-b)) \vee \alpha) \wedge \beta \\ &= ((A(x-a) \vee \alpha) \wedge (A(y-b) \vee \alpha)) \wedge \beta \\ &= (A(0) \vee \alpha) \wedge \beta.\end{aligned}$$

Thus we have $(x+y) \oplus A = (a+b) \oplus A$. Also

$$\begin{aligned}(A([x,y]-[a,b]) \vee \alpha) \wedge \beta &= (A([x-a,y]+[a,y-b]) \vee \alpha) \wedge \beta \\ &\geq (((A([x-a,y]) \vee \alpha) \wedge \beta) \wedge ((A([a,y-b]) \vee \alpha) \wedge \beta)) \vee \alpha) \wedge \beta \\ &\geq ((((A(x-a) \vee \alpha) \wedge \beta) \wedge ((A(y-b) \vee \alpha) \wedge \beta)) \vee \alpha) \wedge \beta \\ &= (A(0) \vee \alpha) \wedge \beta.\end{aligned}$$

So we have $[x,y] \oplus A = [a,b] \oplus A$. □

**Theorem 4.7** Let $A$ be a $(\alpha, \beta)$ fuzzy ideal of $g$, then $g/A$ is a Lie algebra. Where $g/A = \{a \oplus A | \ a \in g\}$,

$(x \oplus A) + (y \oplus A) := (x+y) \oplus A$

$[(x \oplus A), (y \oplus A)] := [x,y] \oplus A$

$\alpha(x \oplus A) := \alpha x \oplus A$.

*Proof.* Clearly. □

**Theorem 4.8** Let $A$ be a $(\alpha, \beta)$ fuzzy ideal of $g$. Then $g/A_{\alpha\beta} \cong g/A$. Where $A_{\alpha\beta} = \{x \in g \mid (A(x) \vee \alpha) \wedge \beta = (A(0) \vee \alpha) \wedge \beta\}$.

*Proof.* Let $f : g \longrightarrow g/A$, $f(x) = x \oplus A$, for all $x \in g$. Then we can easily prove that $f$ is a surjective Lie homomorphism from $g$ to $g/A$. And

$$\begin{aligned}\mathrm{Ker} f &= \{x \in g | f(x) = 0 \oplus A\} \\ &= \{x \in g | x \oplus A = 0 \oplus A\} \\ &= \{x \in g | (A(x) \vee \alpha) \wedge \beta \\ &= (A(0) \vee \alpha \wedge \beta\} = A_{\alpha\beta}.\end{aligned}$$

So $g/A_{\alpha\beta} \cong g/A$. □

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# SOME NECESSARY CONDITIONS FOR THE EXISTENCE OF A FINITE OSBORN LOOP WITH TRIVIAL NUCLEUS

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## Abstract

In this work, the 2005 open problem "Does there exist a proper Osborn loop with a trivial nucleus?" of Kinyon for finite Osborn loops is expressed in terms of the orders of the nucleus, 2nd Bryant-Schneider and automorphism groups of the loop. Furthermore, some necessary conditions(based on the orders of the 1st Bryant-Schneider, 2nd Bryant-Schneider and automorphism groups) for the existence of a universal(left, right universal) Osborn loop with trivial nucleus are deduced. Also, some sufficient conditions(based on the orders of the 1st, 2nd Bryant-Schneider and automorphism groups) for the non-existence of a universal(left, right universal) Osborn loop with trivial nucleus are deduced. It is observed that these results altogether can be used to peep through the windows of the rooms of solutions to the Phillips' open problem and Doro's conjecture.

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# 1 Introduction

A loop is called an Osborn loop if it obeys the identity below.

$$OS_0 \ : \ x(yz \cdot x) = x(yx^{\lambda} \cdot x) \cdot zx \tag{1}$$

For a comprehensive introduction to Osborn loops and its universality, and a detailed literature review on it, readers should check Jaiyéọlá [6, 5], Jaiyéọlá and Adéníran [8, 7], and Jaiyéọlá, Adéníran and Sòlárìn [9]. In this present paper, we shall follow the style and notations used in Jaiyéọlá and Adéníran [7], and Jaiyéọlá, Adéníran and Sòlárìn [9]. Some concepts and notions, and results which will be introduced and stated here are those that were not defined or stated in Jaiyéọlá and Adéníran [7], and Jaiyéọlá , Adéníran and Sòlárìn [9].

In this work, the 2005 open problem "Does there exist a proper Osborn loop with a trivial nucleus?" of Kinyon [10] for finite Osborn loops is expressed in terms of the orders of the nucleus, $2^{nd}$ Bryant-Schneider and automorphism groups of the loop. Furthermore, some necessary conditions(based on the orders of the $1^{st}$ Bryant-Schneider, $2^{nd}$ Bryant-Schneider and automorphism groups) for the existence of a universal(left, right universal) Osborn loop with trivial nucleus are deduced. Also, some sufficient

conditions(based on the orders of the 1st, 2nd Bryant-Schneider and automorphism groups) for the non-existence of a universal(left, right universal) Osborn loop with trivial nucleus are deduced. It is observed that these results altogether can be used to peep through the windows of the rooms of solutions to the Phillips' open problem and Doro's conjecture.

**Theorem 1.1** *Let $(G,\cdot)$ be a "certain" loop where "certain" is an isomorphic invariant property. $(G,\cdot)$ is a universal "certain" loop if and only if every $f,g$-principal isotope $(G,*)$ of $(G,\cdot)$ has the "certain" loop property.*

**Definition 1.1** *(Robinson [11])*
*Let $(G,\cdot)$ be a loop.*

1. *A mapping $\theta \in SYM(G,\cdot)$ is a right special map for $G$ means that there exist $f \in G$ so that $(\theta, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)$.*

2. *A mapping $\theta \in SYM(G,\cdot)$ is a left special map for $G$ means that there exist $g \in G$ so that $(\theta R_g^{-1}, \theta, \theta) \in AUT(G,\cdot)$.*

3. *A mapping $\theta \in SYM(G,\cdot)$ is a special map for $G$ means that there exist $f,g \in G$ so that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)$.*

From Definition 1.1, it can be observed that $\theta$ is a left or right special map for a loop $(G,\cdot)$ with identity element $e$ if and only if $\theta$ is an isomorphism of $(G,\cdot)$ onto some $e,g$- or $f,e$- principal isotope $(G,\circ)$ of $(G,\cdot)$. Moreso, $\theta$ is a special map for a loop $(G,\cdot)$ if and only if $\theta$ is an isomorphism of $(G,\cdot)$ onto some $f,g$-principal isotope $(G,\circ)$ of $(G,\cdot)$.

Robinson [11] went further to show that if

$$BS(G,\cdot) = \{\theta \in SYM(G,\cdot) \ : \ \exists\, f,g \in G \ \ni \ (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)\}$$

i.e the set of all special maps in a loop, then $BS(G,\cdot) \leq SYM(G,\cdot)$ called the Bryant-Schneider group of the loop $(G,\cdot)$ because its importance and motivation stem from the work of Bryant and Schneider [2]. Since the advent of the Bryant-Schneider group, some studies by Adeniran [1] and Chiboka [4] have been done on it relative to CC-loops and extra loops. This

group will now be called the first Bryant-Schneider group($1^{st}$ BSG) and represented by $BS_1(G,\cdot) = BS_1(G)$ for a loop $(G,\cdot)$. Let

$$BS_2(G,\cdot) = \{\theta \in SYM(G) \ : \ G(a,b) \overset{\theta}{\cong} G(c,d) \text{ for some } a,b,c,d \in G\}.$$

As shown in Bryant and Schneider [2], $BS_2(G,\cdot)$ forms a group for a loop $(G,\cdot)$ and it shall be called the second Bryant-Schneider group($2^{nd}$ BSG) of the loop. It is easy to see that $BS_1(G,\cdot) \leq BS_2(G,\cdot) \leq SYM(G)$. The $2^{nd}$ BSG will be more useful than the $1^{st}$ BSG in this study. This is as a result of some mappings that are in the $2^{nd}$ BSG and not in the $1^{nd}$ BSG.

**Results of Bryant and Schneider [2]**

**Theorem 1.2** *Let $(Q,\cdot)$ be a finite quasigroup such that $\left|N_\mu\left(Q(a,b,\circ)\right)\right| = k$. If $\theta \in BS_2(Q,\cdot)$, then $Q(a,b,\circ)$ is isomorphic under $\theta$ to $k$ different principal loop isotopes of $(Q,\cdot)$.*

**Theorem 1.3** *If $(Q,\cdot)$ is a finite quasigroup, then the number of principal loop isotopes of $(Q,\cdot)$ isomorphic to $Q(a,b)$ is $\dfrac{|BS_2(Q,\cdot)|\ \left|N_\mu\left(Q(a,b)\right)\right|}{|AUM(Q(a,b))|}$.*

**Theorem 1.4** *Let $(Q,\cdot)$ be a finite loop of order $n$. All loop isotopes of $(Q,\cdot)$ are isomorphic if and only if $\dfrac{|BS_2(Q,\cdot)|\ \left|N_\mu(Q,\cdot)\right|}{|AUM(Q,\cdot)|} = n^2$.*

**Results of Robinson [11]** Let $(Q,\cdot)$ be a loop and $ROB(Q,\cdot) = ROB(Q)$ be the set of autotopisms $\mathfrak{R} = (\delta R_g^{-1}, \delta L_f^{-1}, \delta)$ for $f,g, \in Q$. The author observed that $ROB(Q) \leq AUT(Q)$ and we shall call it he Robinson group(ROBG) of a loop. Furthermore, he mentioned that the mapping $\Theta \ : \ ROB(Q,\cdot) \longrightarrow BS_1(Q,\cdot)$ defined by $\Theta \ : \ (\delta R_g^{-1}, \delta L_f^{-1}, \delta) \longrightarrow \delta$ is an homomorphism and proved the following result about its kernel.

**Theorem 1.5** *Let $(Q,\cdot)$ be a loop, let the mapping $\Theta \ : \ ROB(Q,\cdot) \longrightarrow BS_1(Q,\cdot)$ and let*

$$\mathfrak{L}(Q,\cdot) = \mathfrak{L}(Q) = \{(a,b) \in Q \times Q \mid (Q,a,b) \cong (Q,\cdot)\}.$$

*Then, $|N_\mu(Q)| = |\ker\Theta|$, $|ROB(Q)| = |BS_1(Q)||N_\mu(Q)| = |\mathfrak{L}(Q)||AUM(Q)|$.*

# 2 Main Results

**Theorem 2.1** *Let $(Q,\cdot,\backslash,/)$ be a loop, $(Q,\circ)$ an arbitrary principal isotope of $(Q,\cdot)$ and $(Q,*)$ some principal isotopes of $(Q,\cdot)$. Let $\phi(x,u,v) = (u\backslash([(uv)/(u\backslash(xv))]v))$ and $\gamma(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ for all $x,u,v \in Q$, then $(Q,\cdot,\backslash,/)$ is a universal Osborn loop if and only if the composition*

$$(Q,\cdot) \xrightarrow[\text{principal isotopism}]{(R_{\phi(x,u,v)},L_u,I)} (Q,*) \xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)} (Q,\circ) \xrightarrow[\text{principal isotopism}]{(\mathbb{R}_v,\mathbb{L}_x,I)} (Q,\cdot) \quad (2)$$

*holds.*

**Proof**
Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be an Osborn loop with any arbitrary principal isotope $\mathfrak{Q} = (Q,\blacktriangle,\nwarrow,\nearrow)$ such that

$$x\blacktriangle y = xR_v^{-1}\cdot yL_u^{-1} = (x/v)\cdot(u\backslash y)\ \forall\ u,v \in Q.$$

If $\mathcal{Q}$ is a universal Osborn loop then, $\mathfrak{Q}$ is an Osborn loop. $\mathfrak{Q}$ obeys identity $OS_0$ implies

$$x\blacktriangle[(y\blacktriangle z)\blacktriangle x] = \{x\blacktriangle[(y\blacktriangle x^{\lambda'})\blacktriangle x]\}\blacktriangle(z\blacktriangle x) \quad (3)$$

where $x^{\lambda'} = xJ_{\lambda'}$ is the left inverse of $x$ in $\mathfrak{Q}$. The identity element of the loop $\mathfrak{Q}$ is $uv$. So,

$$x\blacktriangle y = xR_v^{-1}\cdot yL_u^{-1} \text{ implies } y^{\lambda'}\blacktriangle y = y^{\lambda'}R_v^{-1}\cdot yL_u^{-1} = uv \text{ implies}$$

$$y^{\lambda'}R_v^{-1}R_{yL_u^{-1}} = uv \text{ implies } yJ_{\lambda'} = (uv)R_{yL_u^{-1}}^{-1}R_v = (uv)R_{(u\backslash y)}^{-1}$$
$$R_v = [(uv)/(u\backslash y)]v.$$

Thus, using the fact that

$$x\blacktriangle y = (x/v)\cdot(u\backslash y),$$

$\mathfrak{Q}$ is an Osborn loop if and only if

$$(x/v)\cdot u\backslash\{[(y/v)\cdot(u\backslash z)]/v\cdot(u\backslash x)\} = ((x/v)\cdot u\backslash\{[(y/v)(u\backslash([(uv)/(u\backslash x)]v))]/v\cdot (u\backslash x)\})/v\cdot u\backslash[(z/v)(u\backslash x)].$$

Do the following replacements:

$$x' = x/v \Rightarrow x = x'v,\ z' = u\backslash z \Rightarrow z = uz',\ y' = y/v \Rightarrow y = y'v$$

we have

$$x' \cdot u\backslash\{(y'z')/v \cdot [u\backslash(x'v)]\} = (x' \cdot u\backslash\{[y'(u\backslash([(uv)/(u\backslash(x'v))]v))]/v \cdot [u\backslash(x'v)]\})/v \cdot u\backslash[((uz')/v)(u\backslash(x'v))].$$

This is precisely identity $\mathrm{OS}_0'$ below by replacing $x'$, $y'$ and $z'$ by $x$, $y$ and $z$ respectively.

$$\underbrace{x \cdot u\backslash\{(yz)/v \cdot [u\backslash(xv)]\} = (x \cdot u\backslash\{[y(u\backslash([(uv)/(u\backslash(xv))]v))]/v \cdot [u\backslash(xv)]\})/v \cdot u\backslash[((uz)/v)(u\backslash(xv))]}_{\mathrm{OS}_0'}.$$

Writing identity $\mathrm{OS}_0'$ in autotopic form, we will obtain the fact that the triple $\big(\alpha(x,u,v), \beta(x,u,v), \gamma(x,u,v)\big) \in AUT(\mathcal{Q})$ for all $x, u, v \in Q$ where $\alpha(x,u,v) = R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x \mathbb{R}_v$, $\beta(x,u,v) = L_u \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_v$ and $\gamma(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ are elements of $\mathcal{M}\mathrm{ult}(Q)$. The triple

$$\big(\alpha(x,u,v), \beta(x,u,v), \gamma(x,u,v)\big) = \Big(R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\gamma\mathbb{R}_v, L_u\gamma\mathbb{L}_x, \gamma\Big)$$

can be written as the following compositions $\Big(R_{(u\backslash([(uv)/(u\backslash(xv))]v))}, L_u, I\Big)$ $(\gamma,\gamma,\gamma)(\mathbb{R}_v, \mathbb{L}_x, I)$.

Let $(Q,\circ)$ be an arbitrary principal isotope of $(Q,\cdot)$ and $(Q,*)$ a particular principal isotope of $(Q,\cdot)$. Let $\phi(x,u,v) = (u\backslash([(uv)/(u\backslash(xv))]v))$, then the composition above can be expressed as:

$$(Q,\cdot) \xrightarrow[\text{principal isotopism}]{(R_{\phi(x,u,v)}, L_u, I)} (Q,*) \xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)} (Q,\circ) \xrightarrow[\text{principal isotopism}]{(\mathbb{R}_v, \mathbb{L}_x, I)} (Q,\cdot).$$

The proof of the converse is as follows. Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be an Osborn loop. Assuming that the composition in equation (2) holds, then doing the reverse of the proof of necessity, $\big(\alpha(x,u,v), \beta(x,u,v), \gamma(x,u,v)\big) \in AUT(\mathcal{Q})$ for all $x, u, v \in Q$ which means that $\mathcal{Q}$ obeys identity $\mathrm{OS}_0'$ hence, it will be observed that equation (3) is true for any arbitrary $u,v$-principal isotope $\mathfrak{Q} = (Q, \blacktriangle, \nwarrow, \nearrow)$ of $\mathcal{Q}$. So, every $f,g$-principal isotope $\mathfrak{Q}$ of $\mathcal{Q}$ is an Osborn loop. Following Theorem 1.1, $\mathcal{Q}$ is a universal Osborn loop if and only if $\mathfrak{Q}$ is an Osborn loop.

**Theorem 2.2** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite universal Osborn loop. Any arbitrary $x, v$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different principal loop isotopes of $\mathcal{Q}$ under the tri-mapping $\gamma(x,u,v)^{-1} = \mathbb{L}_x L_u \mathbb{R}_{[u\backslash(xv)]} R_v$.*

**Proof**
The proof of this is achieved by using Theorem 2.1 and Theorem 1.2.

**Theorem 2.3** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a universal Osborn loop if and only if any arbitrary $x, v$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different principal loop isotopes of $\mathcal{Q}$ under the tri-mapping $\gamma(x,u,v)^{-1} = \mathbb{L}_x L_u \mathbb{R}_{[u\backslash(xv)]} R_v$.*

**Proof**
The proof of this is achieved by using Theorem 2.1, Theorem 2.2 and Theorem 1.2.

**Corollary 2.1** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL implies any arbitrary $x, v$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different principal loop isotopes of $\mathcal{Q}$ under the tri-mapping $\gamma(x,u,v)^{-1} = \mathbb{L}_x L_u \mathbb{R}_{[u\backslash(xv)]} R_v$.*

**Proof**
Consequence of Theorem 2.3 or Theorem 2.2.

**Theorem 2.4** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a universal Osborn loop if and only if any principal isotope of $\mathcal{Q}$ is isomorphic to $\dfrac{|BS_2(\mathcal{Q})|\ |N(\mathcal{Q})|}{|AUM(\mathcal{Q})|}$ principal loop isotopes of $\mathcal{Q}$.*

**Proof**
Theorem 2.1 and Theorem 1.3 prove this.

**Corollary 2.2** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL implies any principal isotope of $\mathcal{Q}$ is isomorphic to $\dfrac{|BS_2(\mathcal{Q})|\ |N(\mathcal{Q})|}{|AUM(\mathcal{Q})|}$ principal loop isotopes of $\mathcal{Q}$.*

**Proof**
Direct consequence of Theorem 2.4.

**Theorem 2.5** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite Osborn loop of order $n$. $\mathcal{Q}$ is a universal Osborn loop and a G-loop if and only if $|BS_2(\mathcal{Q})|\ |N(\mathcal{Q})| = n^2|AUM(\mathcal{Q})|$.*

**Proof**
Theorem 2.1 and Theorem 1.4 prove this.

**Corollary 2.3** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite extra loop, CC-loop or VD-loop of order $n$. Then, $|BS_2(\mathcal{Q})| = \dfrac{n^2|AUM(\mathcal{Q})|}{|N(\mathcal{Q})|}$.*

**Proof**
Direct consequence of Theorem 2.5.

**Corollary 2.4** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite group of order $n$. Then, $|BS_2(\mathcal{Q})| = n|AUM(\mathcal{Q})|$.*

**Proof**
Direct consequence of Theorem 2.5.

**Corollary 2.5** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite Moufang loop or universal WIPL of order $n$. $\mathcal{Q}$ is a G-loop if and only if $|BS_2(\mathcal{Q})|\ |N(\mathcal{Q})| = n^2|AUM(\mathcal{Q})|$.*

**Proof**
Direct consequence of Theorem 2.5.

## Some necessary conditions for the existence of a universal Osborn loop with trivial nucleus

**Lemma 2.1** *If there exists a finite universal Osborn loop $\mathcal{Q}$ with trivial nucleus that is not a G-loop, then*

1. *$|BS_2(\mathcal{Q})| = |AUM(\mathcal{Q})|$, hence $BS_2(\mathcal{Q}) = BS_1(\mathcal{Q})$;*

   *or*

*2.* $|AUM(\mathcal{Q})||BS_2(\mathcal{Q})|$.

**Proof**
1. and 2. can be reasoned out with Theorem 2.4. The proof of the later part of 1. is as follows. Let $\mathcal{Q}$ be a finite non-associative universal Osborn loop with trivial nucleus that is not necessarily a G-loop. We have shown already that, $|AUM(\mathcal{Q})| = |BS_2(\mathcal{Q})|$. By Theorem 1.5, $|ROB(\mathcal{Q})| = |BS_1(\mathcal{Q})| = |\mathfrak{L}(\mathcal{Q})||AUM(\mathcal{Q})|$. So, $|BS_2(\mathcal{Q})|\Big||BS_1(\mathcal{Q})|$. By Lagrange's theorem, $|BS_1(\mathcal{Q})|\Big||BS_2(\mathcal{Q})|$, thus $|BS_1(\mathcal{Q})| = |BS_2(\mathcal{Q})|$, which implies that $BS_1(\mathcal{Q}) = BS_2(\mathcal{Q})$ since $BS_1(\mathcal{Q}) \leq BS_2(\mathcal{Q})$.

**Lemma 2.2** *There does not exist a finite universal Osborn loop* $\mathcal{Q}$ *that is with trivial nucleus if* $|BS_2(\mathcal{Q})|$ *is prime and* $|BS_2(\mathcal{Q})| \neq |AUM(\mathcal{Q})|$ *or* $|AUM(\mathcal{Q})| \nmid |BS_2(\mathcal{Q})|$.

**Proof**
This can be reasoned out with Theorem 2.4 by using proof by contradiction.

**Lemma 2.3** *If there exists a finite universal Osborn loop* $\mathcal{Q}$ *that is also a G-loop with trivial nucleus, then* $|AUM(\mathcal{Q})||BS_2(\mathcal{Q})|$ *and* $\frac{|BS_2(\mathcal{Q})|}{|AUM(\mathcal{Q})|}$ *is a perfect square i.e* $|\mathcal{Q}| = \sqrt{\frac{|BS_2(\mathcal{Q})|}{|AUM(\mathcal{Q})|}}$.

**Proof**
This can be reasoned out with Theorem 2.5.

**Example 2.1** *The smallest Moufang loop* $\mathcal{Q}$ *of order* 12 *is a G-loop and has a trivial nucleus(Chein and Pflugdelder [3]). Hence by Lemma 2.3,* $|BS_2(\mathcal{Q})| = 144\ |AUM(\mathcal{Q})|$.

**Lemma 2.4** *There does not exist a non-trivial finite universal Osborn loop* $\mathcal{Q}$ *that is also a G-loop with trivial nucleus if* $|BS_2(\mathcal{Q})| = |AUM(\mathcal{Q})|$ *or* $|AUM(\mathcal{Q})| \nmid |BS_2(\mathcal{Q})|$.

**Proof**
This can be reasoned out with Theorem 2.5.

**Remark 2.1** *In Example 2.1, if the condition(s) in Lemma 2.4 are true, then there will be ambiguity.*

**Lemma 2.5** *There is no non-trivial group with a trivial nucleus.*

**Corollary 2.6** *If there exist a finite non-associative universal Osborn loop that is a G-loop with trivial nucleus, then* $BS_2(\mathcal{Q}) = BS_1(\mathcal{Q})$.

**Proof**
Let $\mathcal{Q}$ be a finite non-associative universal Osborn loop with trivial nucleus that is a G-loop. According to Lemma 2.3, $|AUM(\mathcal{Q})|||BS_2(\mathcal{Q})|$ and $|\mathcal{Q}| = \sqrt{\frac{|BS_2(\mathcal{Q})|}{|AUM(\mathcal{Q})|}}$. By Theorem 1.5, $|BS_1(\mathcal{Q})| = |\mathfrak{L}(\mathcal{Q})||AUM(\mathcal{Q})|$ and by Lagrange's theorem, $|BS_2(\mathcal{Q})| = |BS_1(\mathcal{Q})|[BS_2(\mathcal{Q}) : BS_1(\mathcal{Q})]$, so that $|BS_2(\mathcal{Q})| = |\mathfrak{L}(\mathcal{Q})||AUM(\mathcal{Q})|[BS_2(\mathcal{Q}) : BS_1(\mathcal{Q})]$. Since $\mathcal{Q}$ is a G-loop, $|\mathfrak{L}(\mathcal{Q})| = |\mathcal{Q}|^2$, thus, $|\mathcal{Q}| = \sqrt{\frac{|BS_2(\mathcal{Q})|}{|AUM(\mathcal{Q})|[BS_2(\mathcal{Q}) : BS_1(\mathcal{Q})]}}$. Now $\sqrt{\frac{|BS_2(\mathcal{Q})|}{|AUM(\mathcal{Q})|}} = \sqrt{\frac{|BS_2(\mathcal{Q})|}{|AUM(\mathcal{Q})|[BS_2(\mathcal{Q}) : BS_1(\mathcal{Q})]}}$ implies that $[BS_2(\mathcal{Q}) : BS_1(\mathcal{Q})] = 1$ which means that $|BS_2(\mathcal{Q})| = |BS_1(\mathcal{Q})|$, hence $BS_2(\mathcal{Q}) = BS_1(\mathcal{Q})$.

**Example 2.2** *The smallest Moufang loop* $\mathcal{Q}$ *of order* 12 *is a G-loop and has a trivial nucleus(Chein and Pflugdelder [3]). Hence by Corollary 2.6,* $BS_2(\mathcal{Q}) = BS_1(\mathcal{Q})$.

**Theorem 2.6** *Let* $(Q, \cdot, \backslash, /)$ *be a loop,* $(Q, \circ)$ *an arbitrary principal isotope of* $(Q, \cdot)$ *and* $(Q, *)$ *some left principal isotopes of* $(Q, \cdot)$. *Let* $\phi(x, v) = ([v/(xv)]v)$ *and* $\gamma(x, v) = \mathbb{R}_v R_{(xv)} \mathbb{L}_u L_x$ *for all* $x, v \in Q$, *then* $(Q, \cdot, \backslash, /)$ *is a left universal Osborn loop if and only if the composition*

$$(Q, \cdot) \xrightarrow[\text{left principal isotopism}]{(R_{\phi(x,v)}, I, I)} (Q, *) \xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)} (Q, \circ) \xrightarrow[\text{principal isotopism}]{(\mathbb{R}_v, \mathbb{L}_x, I)} (Q, \cdot) \tag{4}$$

*holds.*

**Proof**
The method of the proof of this theorem is similar to the method used to prove Theorem 2.1 by just using the arbitrary left principal isotope $\mathfrak{Q} = (Q, \blacktriangle, \nwarrow, \nearrow)$ such that

$$x \blacktriangle y = xR_v^{-1} \cdot y = (x/v) \cdot y \; \forall \; v \in Q.$$

In the process of the proof, it will be observed that a loop $\mathcal{Q} = (Q, \cdot, \backslash, /)$ is a left universal Osborn loop if and only if it obeys the identity

$$\underbrace{x \cdot [(y \cdot zv)/v \cdot (xv)] = (x \cdot \{[y([v/(xv)]v)]/v \cdot (xv)\})/v \cdot [z \cdot xv]}_{\mathrm{OS}_0^{\lambda}}.$$

Writing identity $\mathrm{OS}_0^{\lambda}$ in autotopic form, we can conclude that $\mathcal{Q}$ is a left universal Osborn loop if and only if the triple $\big(\alpha(x,v), \beta(x,v), \gamma(x,v)\big) \in AUT(Q)$ for all $x, v \in Q$ where $\alpha(x,v) = R_{([v/(xv)]v)}\mathbb{R}_v R_{[xv]} L_x \mathbb{R}_v$, $\beta(x,v) = \mathbb{R}_v R_{[xv]} \mathbb{L}_v$ and $\gamma(x,v) = \mathbb{R}_v R_{[xv]} L_x$ are elements of $\mathcal{M}\mathrm{ult}(Q)$. Breaking this into compositions like we did in the proof of Theorem 2.1, we shall get equation (4).

**Theorem 2.7** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite left universal Osborn loop. Any arbitrary $x, v$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different left principal loop isotopes of $\mathcal{Q}$ under the bi-mapping $\gamma(x,v)^{-1} = \mathbb{L}_x \mathbb{R}_{(xv)} R_v$.*

**Proof**
The proof of this is achieved by using Theorem 2.6 and Theorem 1.2.

**Theorem 2.8** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a left universal Osborn loop if and only if any arbitrary $x, v$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different left principal loop isotopes of $\mathcal{Q}$ under the bi-mapping $\gamma(x,v)^{-1} = \mathbb{L}_x \mathbb{R}_{(xv)} R_v$.*

**Proof**
The proof of this is achieved by using Theorem 2.6, Theorem 2.7 and Theorem 1.2.

**Corollary 2.7** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL implies any arbitrary $x, v$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different left principal loop isotopes of $\mathcal{Q}$ under the bi-mapping $\gamma(x,v)^{-1} = \mathbb{L}_x \mathbb{R}_{(xv)} R_v$.*

**Proof**
Consequence of Theorem 2.8 or Theorem 2.7.

**Theorem 2.9** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a left universal Osborn loop if and only if any principal isotope of $\mathcal{Q}$ is isomorphic to $\frac{|BS_2(\mathcal{Q})|\ |N(\mathcal{Q})|}{|AUM(\mathcal{Q})|}$ left principal loop isotopes of $\mathcal{Q}$.*

**Proof**
Theorem 2.6 and Theorem 1.3 prove this.

**Corollary 2.8** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL implies any principal isotope of $\mathcal{Q}$ is isomorphic to $\frac{|BS_2(\mathcal{Q})|\ |N(\mathcal{Q})|}{|AUM(\mathcal{Q})|}$ left principal loop isotopes of $\mathcal{Q}$.*

**Proof**
Direct consequence of Theorem 2.9.

## Some necessary conditions for the existence of a left universal Osborn loop with trivial nucleus

**Lemma 2.6** *If there exists a finite left universal Osborn loop $\mathcal{Q}$ with trivial nucleus that is not a G-loop, then*

1. *$|BS_2(\mathcal{Q})| = |AUM(\mathcal{Q})|$, hence $BS_2(\mathcal{Q}) = BS_1(\mathcal{Q})$;*

   *or*

2. *$|AUM(\mathcal{Q})|\big||BS_2(\mathcal{Q})|$.*

**Proof**
1. and 2. can be reasoned out with Theorem 2.9. The proof of the later part of 1. is as follows. Let $\mathcal{Q}$ be a finite non-associative universal Osborn loop with trivial nucleus that is not necessarily a G-loop. We have shown already that, $|AUM(\mathcal{Q})| = |BS_2(\mathcal{Q})|$. By Theorem 1.5, $|ROB(\mathcal{Q})| = |BS_1(\mathcal{Q})| = |\mathfrak{L}(\mathcal{Q})||AUM(\mathcal{Q})|$. So, $|BS_2(\mathcal{Q})|\Big||BS_1(\mathcal{Q})|$. By Lagrange's theorem, $|BS_1(\mathcal{Q})|\Big||BS_2(\mathcal{Q})|$, thus $|BS_1(\mathcal{Q})| = |BS_2(\mathcal{Q})|$, which implies that $BS_1(\mathcal{Q}) = BS_2(\mathcal{Q})$ since $BS_1(\mathcal{Q}) \leq BS_2(\mathcal{Q})$.

**Lemma 2.7** *There does not exist a finite left universal Osborn loop $\mathcal{Q}$ that is with trivial nucleus if $|BS_2(\mathcal{Q})|$ is prime and $|BS_2(\mathcal{Q})| \neq |AUM(\mathcal{Q})|$ or $|AUM(\mathcal{Q})| \not| |BS_2(\mathcal{Q})|$.*

**Proof**
This can be reasoned out with Theorem 2.9 by using proof by contradiction.

**Theorem 2.10** *Let $(Q, \cdot, \backslash, /)$ be a loop, $(Q, \circ)$ an arbitrary right principal isotope of $(Q, \cdot)$ and $(Q, *)$ some principal isotopes of $(Q, \cdot)$. Let $\phi(x, u) = (u\backslash[u/(u\backslash x)])$ and $\gamma(x, u) = R_{[u\backslash x]}\mathbb{L}_u L_x$ for all $x, u \in Q$, then $(Q, \cdot, \backslash, /)$ is a right universal Osborn loop if and only if the composition*

$$(Q,\cdot) \xrightarrow[\text{principal isotopism}]{(R_{\phi(x,u)},L_u,I)} (Q,*) \xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)} (Q,\circ) \xrightarrow[\text{right principal isotopism}]{(I,\mathbb{L}_x,I)} (Q,\cdot) \tag{5}$$

*holds.*

**Proof**
The method of the proof of this theorem is similar to the method used to prove Theorem 2.1 by just using the arbitrary right principal isotope $\mathfrak{Q} = (Q, \blacktriangle, \nwarrow, \nearrow)$ such that

$$x \blacktriangle y = x \cdot yL_u^{-1} = x \cdot (u\backslash y) \ \forall \ u \in Q.$$

In the process of the proof, it will be observed that a loop $\mathcal{Q} = (Q, \cdot, \backslash, /)$ is a right universal Osborn loop if and only if it obeys the identity

$$\underbrace{(ux) \cdot u\backslash\{yz \cdot x\} = ((ux) \cdot u\backslash\{[y(u\backslash[u/x])] \cdot x\}) \cdot u\backslash[(uz)x].}_{\mathrm{OS}_0^{\rho}}$$

Writing identity $OS_0^{\rho}$ in autotopic form, we can conclude that $\mathcal{Q}$ is a right universal Osborn loop if and only if the triple $\big(\alpha(x,u),\beta(x,u),\gamma(x,u)\big) \in AUT(Q)$ for all $x,u \in Q$ where $\alpha(x,u) = R_{(u\backslash[u/(u\backslash x)])}R_{[u\backslash x]}\mathbb{L}_u L_x$, $\beta(x,u) = L_u R_{[u\backslash x]}$ and $\gamma(x,u) = R_{[u\backslash x]}\mathbb{L}_u L_x$ are elements of $\mathcal{M}\text{ult}(Q)$. Breaking this into compositions like we did in the proof of Theorem 2.1, we shall get equation (5).

**Theorem 2.11** *Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a finite left universal Osborn loop. Any arbitrary $x,e$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different principal loop isotopes of $\mathcal{Q}$ under the bi-mapping $\gamma(x,u)^{-1} = \mathbb{L}_x L_u \mathbb{R}_{[u\backslash x]}$.*

**Proof**
The proof of this is achieved by using Theorem 2.10 and Theorem 1.2.

**Theorem 2.12** *Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a finite loop. $\mathcal{Q}$ is a right universal Osborn loop if and only if any arbitrary $x,e$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different principal loop isotopes of $\mathcal{Q}$ under the bi-mapping $\gamma(x,u)^{-1} = \mathbb{L}_x L_u \mathbb{R}_{[u\backslash x]}$.*

**Proof**
The proof of this is achieved by using Theorem 2.10, Theorem 2.11 and Theorem 1.2.

**Corollary 2.9** *Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a finite loop. $\mathcal{Q}$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL implies any arbitrary $x,e$-principal isotope of $\mathcal{Q}$ is isomorphic to $|N(\mathcal{Q})|$ different principal loop isotopes of $\mathcal{Q}$ under the bi-mapping $\gamma(x,u)^{-1} = \mathbb{L}_x L_u \mathbb{R}_{[u\backslash x]}$.*

**Proof**
Consequence of Theorem 2.12 or Theorem 2.11.

**Theorem 2.13** *Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a finite loop. $\mathcal{Q}$ is a right universal Osborn loop if and only if any right principal isotope of $\mathcal{Q}$ is isomorphic to $\dfrac{|BS_2(\mathcal{Q})|\ |N(\mathcal{Q})|}{|AUM(\mathcal{Q})|}$ principal loop isotopes of $\mathcal{Q}$.*

**Proof**
Theorem 2.10 and Theorem 1.3 prove this.

**Corollary 2.10** *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a finite loop. $\mathcal{Q}$ is a Moufang loop or extra loop or VD-loop or CC-loop or universal WIPL implies any right principal isotope of $\mathcal{Q}$ is isomorphic to $\frac{|BS_2(\mathcal{Q})|\ |N(\mathcal{Q})|}{|AUM(\mathcal{Q})|}$ principal loop isotopes of $\mathcal{Q}$.*

**Proof**
Direct consequence of Theorem 2.13.

**Some necessary conditions for the existence of a right universal Osborn loop with trivial nucleus**

**Lemma 2.8** *If there exists a finite right universal Osborn loop $\mathcal{Q}$ with trivial nucleus that is not be a G-loop, then*

1. $|BS_2(\mathcal{Q})| = |AUM(\mathcal{Q})|$, *hence* $BS_2(\mathcal{Q}) = BS_1(\mathcal{Q})$;

   *or*

2. $|AUM(\mathcal{Q})|\big||BS_2(\mathcal{Q})|$.

**Proof**
1. and 2. can be reasoned out with Theorem 2.13. The proof of the later part of 1. is as follows. Let $\mathcal{Q}$ be a finite non-associative universal Osborn loop with trivial nucleus that is not necessarily a G-loop. We have shown already that, $|AUM(\mathcal{Q})| = |BS_2(\mathcal{Q})|$. By Theorem 1.5, $|ROB(\mathcal{Q})| = |BS_1(\mathcal{Q})| = |\mathfrak{L}(\mathcal{Q})||AUM(\mathcal{Q})|$. So, $|BS_2(\mathcal{Q})|\big||BS_1(\mathcal{Q})|$. By Lagrange's theorem, $|BS_1(\mathcal{Q})|\big||BS_2(\mathcal{Q})|$, thus $|BS_1(\mathcal{Q})| = |BS_2(\mathcal{Q})|$, which implies that $BS_1(\mathcal{Q}) = BS_2(\mathcal{Q})$ since $BS_1(\mathcal{Q}) \leq BS_2(\mathcal{Q})$.

**Lemma 2.9** *There does not exist a finite right universal Osborn loop $\mathcal{Q}$ that is with trivial nucleus if $|BS_2(\mathcal{Q})|$ is prime and $|BS_2(\mathcal{Q})| \neq |AUM(\mathcal{Q})|$ or $|AUM(\mathcal{Q})| \not| |BS_2(\mathcal{Q})|$.*

**Proof**
This can be reasoned out with Theorem 2.13 by using proof by contradiction.

# 3 Concluding Remarks and Future Studies

In this work, we have been able to express the open problem of Kinyon for finite Osborn loops with Theorem 2.3(Theorem 2.8, Theorem 2.12) and Theorem 2.4(Theorem 2.9, Theorem 2.13) in terms of the orders of the nucleus, $2^{nd}$ Bryant-Schneider and automorphism groups of the loop.

Furthermore, we have been able to deduce some necessary conditions (based on the orders of the $1^{st}$ Bryant-Schneider, $2^{nd}$ Bryant-Schneider and automorphism groups) for the existence of a universal(left, right universal) Osborn loop with trivial nucleus. Also, some sufficient conditions(based on the orders of the $1^{st}$, $2^{nd}$ Bryant-Schneider and automorphism groups) for the non-existence of a universal(left, right universal) Osborn loop with trivial nucleus were deduced.

Altogether, these results can be used to peep through the windows of the rooms of solutions to two other open problems of Mile High Conference on Quasigroups, Loops, and Non-associative Systems(since they both have to do with a universal Osborn loop with trivial nucleus). Namely

**Phillips' Problem** Is there a Moufang loop of odd order with trivial nucleus?

**Doro's Conjecture** Does a Moufang loop with trivial nucleus necessarily have normal commutant?

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# ON QUASI CONFORMAL CURVATURE TENSOR IN A RIEMANNIAN MANIFOLD

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## Abstract

In this paper, we have shown that a quasi conformally flat manifold is an Einstein manifold of constant curvature, provided that $a + b(n-2) \neq 0$, where $a, b$ are suitable constants and vice-versa. The necessary and sufficient condition for a Riemannian manifold to be quasi conformally conservative has also been established. Some other interesting results have also been established.

**Keywords and Phrases:** Riemannian manifold, curvature tensor, quasi conformal curvature tensor, Einstein manifold.

**AMS Subject Classification:** 53A30.

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# 1 Preliminaries

The quasi conformal curvature tensor $\bar{C}$ on a Riemannian manifold $(M^n, g)$ of type (1,3) is defined as [1]

$$\bar{C}(X,Y,Z) = aR(X,Y,Z) + b\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\big]$$
$$-\frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)[g(Y,Z)X - g(X,Z)Y] \tag{1.1}$$

where $a, b$ are constants such that $a, b \neq 0$, $R$ curvature tensor, $S$ Ricci tensor and $r$ is the scalar curvature.

If $Q$ is the symmetric endomorphism of tangent space at any point of the manifold corresponding to the Ricci tensor $S$, then

$$S(X,Y) = g(QX,Y) \tag{1.2}$$

If $\bar{C} = 0$, $M^n$ will be called quasi conformally flat manifold. For such manifold, we have in view of the equation (1.1)

$$aR(X,Y,Z) + b\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\big]$$
$$-\frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)[g(Y,Z)X - g(X,Z)Y] = 0 \tag{1.3}$$

Taking inner product with $W$ in equation (1.3), we get

$$a'R(X,Y,Z,W)+b\big[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(Y,Z)S(X,W)-g(X,Z)S(Y,W)\big]$$
$$-\frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] = 0 \tag{1.4}$$

where $'R(X,Y,Z,W) = g(R(X,Y,Z),W)$.

Let $\{e_i\}$, $i = 1, 2, \cdots, n$ be the orthonormal basis of tangent space at any point of the manifold.

Put $X = W = e_i$ in equation (1.4) and taking summation over $i$, $1 \leq i \leq n$ we get

$$[a + b(n-2)]\Big[S(Y,Z) - \frac{r}{n}g(Y,Z)\Big] = 0 \tag{1.5}$$

If $a + b(n-2) \neq 0$, then from equation (1.5) we get

$$S(Y,Z) = \frac{r}{n}g(Y,Z). \tag{1.6}$$

Thus $M^n$ becomes an Einstein manifold.

With the help of equations (1.4) and (1.6) we get

$$'R(X,Y,Z,W) = \frac{r}{n(n-1)}\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \quad \text{for } a \neq 0 \qquad (1.7)$$

Thus we get the following theorem

**Theorem 1.1** *A quasi conformally flat manifold is an Einstein manifold of constant curvature, provided* $a + b(n-2) \neq 0$.

From equation (1.1) we have

$$'\bar{C}(X,Y,Z,W) = a'R(X,Y,Z,W) + b\big[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + S(X,W)g(Y,Z)$$

$$-S(Y,W)g(X,Z)\big] - \frac{r}{n}\Big(\frac{a}{n-1} + 2b\Big)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \qquad (1.8)$$

where $'\bar{C}(X,Y,Z,W) = g(C(X,Y,Z),W)$.

Now (1.8) can be written as follows

$$'\bar{C}(X,Y,Z,W) = a\Big['R(X,Y,Z,W) - \frac{r}{n(n-1)}\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big\}\Big]$$

$$+b\Big[g(X,W)\Big\{S(Y,Z) - \frac{2r}{n}g(Y,Z)\Big\} - g(Y,W)\Big\{S(X,Z) - \frac{2r}{n}g(X,Z)\Big\}$$

$$+S(X,W)g(Y,Z) - S(Y,W)g(X,Z)\Big]. \qquad (1.9)$$

Since manifold is of constant curvature, it is an Einstein manifold. Hence in view of the equation (1.9) we get $'\bar{C}(X,Y,Z,W) = 0$.

Thus we have the following theorem:

**Theorem 1.2** *A manifold* $(M^n, g)$ *of constant curvature is quasi conformally flat.*

## 2 Quasi Conformally Conservative $(M^n, g)$

Here we assume that

$$div\bar{C} = 0 \tag{2.1}$$

where $div$ denotes divergence.

Differentiating equation (1.1) covariently, we get

$$(\nabla_\nu \bar{C})(X,Y,Z) = a(\nabla_\nu R)(X,Y,Z)+b[(\nabla_\nu S)(Y,Z)X-(\nabla_\nu S)(X,Z)Y+g(Y,Z)(\nabla_\nu Q)X$$
$$-g(X,Z)(\nabla_\nu Q)Y] - \frac{(\nabla_v r)[a+2b(n-1)]}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

By contracting the above equation we get

$$(div\bar{C})(X,Y,Z) = a(divR)(X,Y,Z)+b[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)+g(Y,Z)(divQ)X$$
$$-g(X,Z)(divQ)Y] - \frac{[a+2b(n-1)]}{n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{2.2}$$

But we know that

$$(divR)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$$

Hence (2.2) gives

$$(div\bar{C})(X,Y,Z) = (a+b)\big[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\big] + b\big[g(Y,Z)(divQ)X$$
$$-g(X,Z)(divQ)Y\big] - \frac{[a+2b(n-1)]}{n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{2.3}$$

If $a+b=0$, $divQ=0$ then from (2.3) we get

$$(div\bar{C})(X,Y,Z) = \frac{-[a+2b(n-1)]}{n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{2.4}$$

If $(div\bar{C}) = 0$ then from above equation

$$g(Y,Z)dr(X) - g(X,Z)dr(Y) = 0$$

The above equation shows that $r$ is constant. Again if $r$ is constant then from (2.4), we get

$$(div\bar{C})(X,Y,Z) = 0.$$

Thus we get the following theorem:

**Theorem 2.1** *If in a Riemannian manifold $(M^n, g)$, the quasi conformal curvature tensor is such that $a+b=0$, $divQ=0$ then the manifold is quasi conformally conservative iff the scalar curvature is constant.*

If $a+b\neq 0$ and $divQ=0$, then from equation (2.3) we have

$$\frac{(div\bar{C})(X,Y,Z)}{(a+b)} = \big[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)\big]+\frac{b}{a+b}\big[g(Y,Z)(divQ)X$$
$$-g(X,Z)(divQ)Y\big]-\frac{1}{n}\left(\frac{a+2b(n-1)}{(a+b)(n-1)}\right)[g(Y,Z)dr(X)-g(X,Z)dr(Y)]. \qquad (2.5)$$

From the above equation, we get the following theorem:

**Theorem 2.2** *In a Riemannian manifold $(M^n, g)$, the quasi conformal curvature tensor is such that $a+b\neq 0$, $divQ\neq 0$, then the manifold is quasi conformally conservative iff*

$$\big[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)\big]=\frac{1}{n}\left(\frac{a+2b(n-1)}{(a+b)(n-1)}\right)[g(Y,Z)dr(X)$$
$$-g(X,Z)dr(Y)\big]-\frac{b}{a+b}\big[g(Y,Z)(divQ)(X)-g(X,Z)(divQ)(Y)\big]$$

If $a+b=0$ and $divQ\neq 0$, then from equation (2.3) we have

$$\frac{(div\bar{C})(X,Y,Z)}{b}=\big[g(Y,Z)(divQ)X-g(X,Z)(divQ)Y\big]-\frac{1}{n}\left(\frac{a+2b(n-1)}{b(n-1)}\right)$$
$$-\big[g(Y,Z)dr(X)-g(X,Z)dr(Y)\big]. \qquad (2.6)$$

From the above equation, we get the following theorem:

**Theorem 2.3** *In a Riemannian manifold $(M^n, g)$, the quasi conformal curvature tensor is such that $a+b=0$, $divQ\neq 0$, then the manifold is quasi conformally conservative iff*

$$\big[g(Y,Z)(divQ)(X)-g(X,Z)(divQ)(Y)\big]$$
$$=\frac{1}{n}\left(\frac{a+2b(n-1)}{b(n-1)}\right)\big[g(Y,Z)dr(X)-g(X,Z)dr(Y)\big]$$

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# FUNDAMENTALLY VERY THIN $H_v$-STRUCTURES

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**Abstract**

We introduce and study a new class of $H_v$-structures called Fundamentally Very Thin. This is an extension of the well known class of the very thin hyperstructures. We present some applications of these hyperstructures.

**Key words:** Hyperstructures, $H_v$-structures, hopes, $\partial$-hopes.

**AMS Subject Classification:** 20N20, 16Y99.

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## 1. FUNDAMENTAL DEFINITIONS AND THE FUNDAMENTAL RELATIONS

We deal with hyperstructures called *$H_v$-structures* introduced in 1990 [12], which satisfy the *weak axioms* where the non-empty intersection replaces the equality.

Some basic definitions are the following:

In a set H equipped with a hyperoperation (abbreviation *hyperoperation=**hope***)

$$\cdot: H\times H \to \mathcal{P}(H)-\{\varnothing\},$$

we abbreviate by

*WASS* the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and by

*COW* the *weak commutativity*: $xy\cap yx\neq\varnothing$, $\forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called ***$H_v$-semigroup*** if it is *WASS,* it is called ***$H_v$-group*** if it is reproductive $H_v$-semigroup, i.e. $xH=Hx=H$, $\forall x\in H$.

***Motivation.*** We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934 [9], states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an $H_v$-group. This is the motivation to introduce the $H_v$-structures [12,14].

In an $H_v$-semigroup the *powers* of an element $h\in H$ are defined as follows: $h^1=\{h\}$, $h^2=h\cdot h,\ldots$ , $h^n=h\circ h\circ\ldots\circ h$, where $(\circ)$ denotes the *n-ary circle hope,* i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An $H_v$-semigroup $(H,\cdot)$ is called *cyclic of period s*, if there exists an element g, called *generator*, and a natural number n, the minimum one, such that $H= h^1\cup h^2\cup\ldots\cup h^s$. Analogously the cyclicity for the infinite period is defined [11,14]. If there is an element h and a natural number s, the minimum one, such that $H=h^s$, then $(H,\cdot)$ is called *single-power cyclic of period s*.

In an a similar way more complicated hyperstructures can be defined: $(R,+,\cdot)$ is called ***$H_v$-ring*** if $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\quad \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ be an $H_v$-ring, $(\mathcal{M},+)$ be a COW $H_v$-group and there exists an external hope

$$\cdot : \mathbf{R}\times\mathcal{M} \to \mathcal{P}(\mathcal{M}): (a,x) \to ax$$

such that, $\forall a,b\in R$ and $\forall x,y\in\mathcal{M}$ we have

$$a(x+y)\cap(ax+ay) \neq \varnothing, \quad (a+b)x\cap(ax+bx) \neq \varnothing, \quad (ab)x\cap a(bx) \neq \varnothing,$$

then $\mathcal{M}$ is called an ***$H_v$-module*** over F. In the case of an $H_v$-field F instead of $H_v$-ring R, then the ***$H_v$-vector space*** is defined.

For more definitions and applications on $H_v$-structures one can see the books [2,3,5,14] and survey papers as [4,18,22]

The main tool to study hyperstructures are the *fundamental relations.* In 1970 [8] M. Koscas defined in hypergroups the relation $\beta$ and its transitive closure $\beta^*$. This relation is turned to be very important since its connects the hyperstructures with the corresponding classical ones [2,14]. This relation is also defined in $H_v$-groups as well. T. Vougiouklis [12,14] introduced the definitions of $\gamma^*$ and $\varepsilon^*$, which are defined, in $H_v$-rings and $H_v$-vector spaces, respectively. He also named all these relations $\beta^*$, $\gamma^*$ and $\varepsilon^*$, *Fundamental Relations* because they play very important role to the study of hyperstructures especially in the representation theory of them.

***Definition1.1*** The fundamental relations $\beta^*$, $\gamma^*$ and $\varepsilon^*$, are defined, in $H_v$-groups, $H_v$-rings and $H_v$-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively [12,14,17,18].

Specifying the above motivation we remark: Let $(G,\cdot)$ be a group and $R$ be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an $H_v$-group, therefore we have the quotient $(G/R,\cdot)/\beta^*$ which is a group, the *fundamental* one. Remark that the classes of the fundamental group $(G/R,\cdot)/\beta^*$ are a union of some of the $R$-classes. Otherwise, the $(G/R,\cdot)/\beta^*$ has elements classes of G

where they form a partition which classes are larger than the classes of the original partition $R$.

The way to find the fundamental classes is given by the following:

***Theorem 1.2*** Let $(H,\cdot)$ be an $H_v$-group and denote by $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation $\beta$ in H by setting $x\beta y$ iff $\{x,y\}\subset \boldsymbol{u}$ where $\boldsymbol{u}\in\boldsymbol{U}$. Then $\beta^*$ is the transitive closure of $\beta$.

Analogous theorems for the fundamental relations $\gamma^*$ in $H_v$-rings, $\varepsilon^*$ in $H_v$-modules and $H_v$-vector spaces, are also proved. An element is called *single* if its fundamental class is singleton [14].

Fundamental relations are used for general definitions. Thus, an $H_v$-ring $(R,+,\cdot)$ is called ***$H_v$-field*** if $R/\gamma^*$ is a field. The general definition of an $H_v$-Lie algebra was given in [14,17,18,20,22] as follows:

Let $(\boldsymbol{L},+)$ be an $H_v$-vector space over the field $(\mathbf{F},+,\cdot)$, $\varphi:\mathbf{F}\to\mathbf{F}/\gamma^*$, the canonical map and $\omega_F=\{x\in\mathbf{F}:\varphi(x)=0\}$, where 0 is the zero of the fundamental field $\mathbf{F}/\gamma^*$. Similarly, let $\omega_L$ be the core of the canonical map $\varphi':\boldsymbol{L}\to\boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $L/\varepsilon^*$. Consider the *bracket (commutator) hope*:

$$[\,,\,]:\boldsymbol{L}\times\boldsymbol{L}\to \mathcal{P}(L):(x,y)\to[x,y]$$

then $\boldsymbol{L}$ is an ***$H_v$-Lie algebra*** over $\mathbf{F}$ if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$$

$$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,\ \forall x,x_1,x_2,y,y_1,y_2\in L \text{ and } \lambda_1,\lambda_2\in\mathbf{F}$$

(L2) $[x,x]\cap\omega_L\neq\varnothing,\quad\forall x\in\boldsymbol{L}$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \forall x,y\in\boldsymbol{L}$

This is a general definition thus one can use special cases in order to face problems in applied sciences. Moreover, we see how the weak properties can be defined as the above weak linearity (L1), anti-commutativity (L2) and the Jacobi identity (L3).

Let $(H,\cdot)$, $(H,*)$ be $H_v$-semigroups defined on the same set H. $(\cdot)$ is called ***smaller*** than $(*)$, and $(*)$ ***greater*** than $(\cdot)$, iff there exists an

$$f\in Aut(H,*) \quad \text{such that} \quad xy\subset f(x*y),\ \forall x,y\in H.$$

Then we write $\cdot\leq^*$ and we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called ***$H_b$-structure***.

***Theorem 1.3*** *(The Little Theorem).* Greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

This Theorem leads to a partial order on $H_v$-structures and mainly to a correspondence between hyperstructures and posets. The determination of all $H_v$-groups and $H_v$-rings is very interesting but hard. Fortunately, in this direction there are results as one can see, for example, in the paper [1] how many $H_v$-groups and $H_v$-rings there exist, up to isomorphism, for small sets, for several classes of hyperstructures, of two, three or four elements.

Using the partial ordering with the fundamental relations one can give several definitions to obtain constructions used in several applications. We present some of them [16,17,18]:

Let $(H,\cdot)$ be hypergroupoid. We *remove* $h\in H$, if we take the restriction of $(\cdot)$ in the set $H-\{h\}$. $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$ and h does not appear. $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

$H_v$-structures are used in Representation Theory. Representations of $H_v$-groups can be considered by generalized permutations or by $H_v$-matrices [14]. Representations by generalized permutations can be achieved by using the translations. In the representation theory the single elements are playing a crucial role. ***$H_v$-matrix*** is called a matrix if has entries from an $H_v$-ring. The hyperproduct of $H_v$-matrices is defined in a usual manner. The problem of the $H_v$-matrix representations is the following:

Let $(H,\cdot)$ be $H_v$-group. Find a $H_v$-ring $(R,+,\cdot)$, a set $M_R=\{(a_{ij}) \mid a_{ij}\in R\}$ and a map

$$T: H\to M_R:\ h\mapsto T(h) \quad \text{such that} \quad T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing,\ \forall h_1,h_2\in H.$$

The map T is called ***$H_v$-matrix representation***.

If the $T(h_1h_2)\subset T(h_1)(h_2)$, $\forall h_1,h_2\in H$ is valid, then T is called *inclusion representation*. If $T(h_1h_2)=T(h_1)(h_2)=\{T(h)\,|\,h\in h_1h_2\}$, $\forall h_1,h_2\in H$, then T is called *good representation* and then an induced representation T* for the hypergroup algebra is obtained. If T is one to one and good then it is a *faithful representation*.

In representations of $H_v$-groups by $H_v$-matrices, there are two difficulties: To find an $H_v$-ring and an appropriate set of $H_v$-matrices. Hopes on any type of matrices can be defined [7], they are called *helix hopes* and lead to a new kind of representations.

A large class of $H_v$-structures was introduced in [19] see also [20,21]:

Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to G$ be a map. We define a hope $(\partial)$, called ***theta-hope***, we write ***$\partial$-hope***, on G as follows

$$x\partial y = \{f(x)\cdot y, x\cdot f(y)\},\ \forall x,y\in G.\ (\text{resp. } x\partial y = (f(x)\cdot y)\cup(x\cdot f(y),\ \forall x,y\in G)$$

If $(\cdot)$ is commutative then $\partial$ is commutative. If $(\cdot)$ is *COW*, then $\partial$ is *COW*.

Let $(G,\cdot)$ be a groupoid (resp. hypergroupoid) and $f:G\to \boldsymbol{P}(G)-\{\varnothing\}$ be any multivalued map. We define the $(\partial)$, on G as follows

$$x\partial y = (f(x)\cdot y)\cup(x\cdot f(y),\ \forall x,y\in G.$$

Let $(G,\cdot)$ be a groupoid, $f_i:G\to G$, $i\in I$, be a set of maps on G. The $f_\cup:G\to \boldsymbol{P}(G)$: $f_\cup(x)=\{f_i(x)\,|\,i\in I\}$, is *the union* of $f_i(x)$. We have the *union $\partial$-hope* $(\partial)$, on G if we take $f_\cup(x)$. If we take $\underline{f}\equiv f\cup(id)$, then we have the *b-$\partial$-hope*.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. The basic property is that if $(G,\cdot)$ is a semigroup then for every f, the $(\partial)$ is *WASS*.

Another well known and large class of hopes is given as follows [11,14,6]:

Let $(G,\cdot)$ be a groupoid, then for every $P\subset G$, $P\neq\varnothing$, we define the following hopes called *P-hopes*: for all $x,y\in G$

$\underline{P}$: $x\underline{P}y=(xP)y\cup x(Py)$, $\underline{P}_r$: $x\underline{P}_r y=(xy)P\cup x(yP)$, $\underline{P}_l$: $x\underline{P}_l y=(Px)y\cup P(xy)$.

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures*. The most usual case is if $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a

semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS. If in the set G, more operations are defined then obviously for each one operation several P-hopes can be defined.

In 1989 Corsini and Vougiouklis [14], introduced a method, *the uniting elements method,* to obtain stricter algebraic structures, from given ones, through hyperstructure theory. This method was introduced before $H_v$-structures, but in fact the $H_v$-structures appeared in the procedure. ***The uniting elements method*** is the following: Let G be a structure and d be a property, which is not valid, and suppose that d is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient G/d is an $H_v$-structure. Then quotient of G/d by the fundamental relation $\beta^*$, is a stricter structure $(G/d)\beta^*$ for which d is valid. An application of the uniting elements is when more than one properties are desired.

## 2. THE VERY THIN AND THE FUNDAMENTALLY VERY THIN CONSTRUCTIONS

An extreme class is the following [14]: A class of $H_v$-structures, introduced in [13] (1991) is the following

***Definition 2.1*** An $H_v$-structure is called ***very thin*** iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin $H_v$-structure in a set H there exists a hope $(\cdot)$ and a pair $(a,b)\in H^2$ for which $ab=A$, with $cardA>1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

***Theorem 2.2*** Let $(H,\cdot)$ be a finite very thin $H_v$-group, $cardA=n>1$, and let a,b be the only elements of H such that $ab=A$, with $cardA>1$. Then we have the following two cases:

**(i)** either for all $u\in H-\{a\}$, $ua=a$; in which case:

if $n=2$, then there exists a group $(H,*)$ such that $a*b\in A$ and $x*y=xy$ for all $(x,y)\in H^2-\{(a,b)\}$,

if $n>2$, then $a=b$, $H-\{a\}$ is a group and $A=H$ or $A=H-\{a\}$.

**(ii)** or if there exists an element $u\in H$ such that $u\neq a$ and $ua\neq a$, then there exists an almost-associative group (i.e. the associativity is valid for all triples except the ones which contain the product a*b), such that

$$a*b\in A \text{ and } x*y=xy \text{ for all } (x,y)\in H^2-\{(a,b)\}.$$

For a Proof see the book [14] p.148.

Since there is a partial order on $H_v$-groups, we obtain that there are minimals very thin $H_v$-groups and the rest are greater than the minimals. The types of minimals are described by the following:

***Construction 2.3*** The non-degenerate (not single valued) minimal very thin $H_v$-groups $(H,\cdot)$ are the following:

***Type I.*** The non-degenerate minimal very thin $H_v$-group of Type II, is an $H_b$-group $(H,*)$ where the basic group is $(H,\cdot)$ and there are elements $a,b,v\in H$ with $ab\neq v$, for which the only one hyperproduct is $a*b=\{ab,v\}$.

***Type II.*** The non-degenerate minimal very thin $H_v$-group of Type I, is a hypergroup where there is an element $v\in H$ such that $(H-\{v\},\cdot)$ is a group and $vx=xv=v$, $\forall x\in H-\{v\}$ and $vv=H-\{v\}$.

The Type II appeared first in research but one can see that the Type I is an extremely large class of hyperstructures. Moreover Type I can be applied in more complicated hyperstructures as we can easily see in the following.

The above Construction Type II is referred in $H_v$-groups. The Type I in fact is referred in weaker hyperstructures like the $H_v$-semigroups, therefore can be applied in hyperrings in the multiplicative part as well. In [14] one can see several cases of very thin $H_v$-rings appeared for several classes of hyperrings. The application on $H_v$-vector spaces and $H_v$-algebras in any operation internal or external gives also corresponding non-degenerate minimal very thin $H_v$-structures.

Now we extend the definition of the very thin $H_v$-structures, as follows:

***Definition 2.4*** An $H_v$-structure is called ***Fundamentally Very Thin*** (abbreviate by ***FVT***) if all hopes are greater than the very thin hopes under the assumption that they have the same corresponding fundamental structure.

It is easy to see that this class of hyperstructures is turn to be very large. The difficulty to find the fundamental classes can be faced, in some degree, by finding only some elements of several fundamental classes using several hyperproducts.

Special elements and special properties are still valid in the weak form. More properties weak or strong could appear and stricter hyperstructures can be obtained to be used in applied sciences. The most important situation is that the partition yielded connects the hyperstructure theory to the classical one. For example one can see in the representation theory of $H_v$-groups by $H_v$-matrices the main results are completely related with the ones in the classical theory or the results from the classical theory of representations can be extended in the hyperstructure theory as well. Finally this construction guaranties to avoid degenerate cases as much as possible.

## 3. APPLICATIONS

During last decades hyperstructures seem to have a variety of applications not only in other branches of mathematics but also in many other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics to mention but a few. The hyperstructures theory is closely related to fuzzy theory; consequently, hyperstructures can now be widely applicable in industry and production, too.

In several books and papers, such as [3,5,6,10,14,17,22,23,24] one can find numerous applications.

***Applications 3.1*** *The e-hyperstructures.*

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1999 [10] and they are called *e-hyperfields*. The $H_v$-fields, especially both the above introduced types, can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology. We present in the following the main definitions and results restricted in the $H_v$-structures.

***Definitions.*** A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we normally assume that for every element x, there exists an inverse element $x^{-1}$, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called ***e-hyperfield*** if the following axioms are valid: $(F,+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x = x\cdot 0 = 0$, $\forall x\in F$, there exists a multiplicative scalar unit 1, i.e. $1\cdot x = x\cdot 1 = x$, $\forall x\in F$, for every $x\in F$ there exists a unique inverse $x^{-1}$: $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$. The elements of an e-hyperfield are called ***e-hypernumbers***. In the case that the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Adopting the above introduced construction we present a general construction called *the Main e-Construction* [23]: Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes $(\hookrightarrow)$ as follows:

$$x\hookrightarrow y = \{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\},\text{ and } g_1, g_2,\ldots\in G-\{e\}$$

$g_1, g_2,\ldots$ are not necessarily the same for each pair (x,y). Then $(G,\hookrightarrow)$ becomes an $H_v$-group, in fact is $H_b$-group which contains the $(G,\cdot)$. The $H_v$-group $(G,\hookrightarrow)$ is an e-hypergroup. Moreover, if for each x,y such that $xy=e$, so we have $x\hookrightarrow y=xy$, then $(G,\hookrightarrow)$ becomes a strong e-hypergroup.

***Example.*** Consider the finite quaternion group

$$\boldsymbol{Q} = \{\mathbf{1, -1, i, -i, j, -j, k, -k}\}$$

whose multiplication is given by the table

| · | 1 | -1 | i | -i | j | -j | k | -k |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | -1 | i | -i | j | -j | k | -k |
| -1 | -1 | 1 | -i | I | -j | j | -k | k |
| I | i | -i | -1 | 1 | k | -k | -j | j |
| -i | -i | i | 1 | -1 | -k | k | j | -j |
| j | j | -j | -k | K | -1 | 1 | i | -i |
| -j | -j | j | k | -k | 1 | -1 | -i | i |
| k | k | -k | j | -j | -i | i | -1 | 1 |
| -k | -k | k | -j | J | i | -i | 1 | -1 |

Using this operation one can obtain several hopes which define very interesting e-groups. For example, denoting $\underline{i}=\{i,-i\}$, $\underline{j}=\{j,-j\}$, $\underline{k}=\{k,-k\}$ we may define the (*) hope, which is a FVT by the table:

| * | 1 | -1 | i | -i | j | -j | k | -k |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | -1 | i | -i | j | -j | k | -k |
| -1 | -1 | 1 | -i | I | -j | j | $\underline{k}$ | k |
| i | i | -i | -1 | 1 | k | -k | -j | j |
| -i | -i | $\underline{i}$ | 1 | -1 | -k | k | J | -j |
| j | j | -j | -k | k | -1 | 1 | I | -i |
| -j | -j | j | k | -k | 1 | -1 | -i | i |
| k | k | $\underline{k}$ | $\underline{j}$ | -j | -i | i | -1 | 1 |
| -k | -k | k | -j | j | i | -i | 1 | -1 |

The representation theory and the Lie algebras as well as in hypermatrix theory large classes of e-hyperstructures appear and can offer to Lie-Santilli algebraic theory models to represent their theory.

***Applications 3.2*** *The Type I hyperstructures. Minimal cases.*

As the uniting elements procedure does, the minimal very thin hopes can lead to stricter structures by a quotient. Normally one can apply these procedures with intention to obtain structures with more properties. However, the procedure can be applied with no intension. The only one suggestion is to apply 'small' hopes because the quotients could be degenerate. In order to avoid degenerate cases the very thin case, as well as the FVT, is the suggested one. We present now a construction analogous to the one given in the book [14] p.75.

***Construction:*** Consider the additive group of integers $(\boldsymbol{Z},+)$ and we take the minimal very thin hope defined by enlarging the only one result of the elements 1 and n-1 setting

$$1\oplus(n-1)=\{0,n\} \quad \text{and} \quad x\oplus y = x+y \quad \text{in all the other cases.}$$

One can prove that we have

$$(\boldsymbol{Z},\oplus)/\beta^* \cong (\boldsymbol{Z}_n,+).$$

Therefore the fundamental group is of finite order, which is completely different from the initial $H_v$-group, which is of infinitive order.

Remark that the reason to enlarge only this result instead of the one in the book, where the enlarged result is the one of the elements 0 and n, is because this construction gives an e-construction.

***Applications 3.3*** *Hyperstructures in social sciences: questionnaires.*

In Social Sciences, where questionnaires are used, seem to have hyperstructure theory as an organized devise. An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis. The suggestion is the following [24]:

***Definition.*** *"In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 ——————————————— 1

*The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question".*

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing [24]. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm. The hyperstructure theory, especially minimal construction, gives innovating new suggestions to connect several groups of objects. These suggestions are obtained from properties and special elements inside the hyperstructure.

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# CLASSES OF MORPHISMS VERSUS SIEVE-PRESHEAVES

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## Abstract

In this article for a given category $\mathcal{X}$, we give connections between certain classes of $\mathcal{X}$-morphisms and certain sieve-presheaves on $\mathcal{X}$, i.e., sieve valued presheaves to the quasicategory of preordered classes; and then we set up a one to one correspondence between classes of morphisms in $\mathcal{X}$ satisfying certain conditions and sieve-presheaves on $\mathcal{X}$ that factor through some sub(quasi)categories of preordered classes.

**Keywords:** (complete) preordered or partially ordered class or set, presheaf, sieve.

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## 1. Preliminaries

Classes of morphisms of a category are studied in relation to closure operators, see [2] and [3]. In the present section we introduce and investigate several properties of such classes.

In Section 2, for a given category $\mathcal{X}$, we associate to a certain class $\mathcal{M}$ of morphisms in $\mathcal{X}$, a sieve-presheaf $M$ on $\mathcal{X}$, i.e., a sieve-valued functor $M$ from $\mathcal{X}^{op}$ to the quasicategory of partially ordered classes; and to a sieve-presheaf $M$ a class of morphisms $\mathcal{M}$ in $\mathcal{X}$. We give necessary and sufficient conditions on $\mathcal{M}$ so that the corresponding sieve-presheaf $M$ factors through the (quasi)category of partially ordered sets or complete partially ordered classes (or sets).

Finally in Section 3, we give several illustrative and supporting examples.

**Definition 1.1.** Let $\mathcal{X}$ be a category and for an object $x$, $\mathcal{X}_1/x$ be the class of all morphisms to $x$. Define a preorder on $\mathcal{X}_1/x$, by $f \leq g$ if there is a morphism $\alpha$ such that $f = g \circ \alpha$ and let "$\sim$" be the equivalence relation generated by "$\leq$", so that $f \sim g$ if and only if $f \leq g$ and $g \leq f$. For $\mathcal{M} \subseteq \mathcal{X}_1$, the above preorder and equivalence relation on $\mathcal{X}_1/x$ can be passed over to $\mathcal{M}/x$. Also we write $m \sim \mathcal{M}/x$ ($m \sim \mathcal{M}$) if there is $n \in \mathcal{M}/x$ ($n \in \mathcal{M}$) such that $m \sim n$.

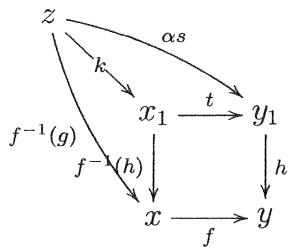
Denoting a pullback of $g$ along $f$ by $f^{-1}(g)$, we have:

**Lemma 1.2.** *Let $f : x \longrightarrow y$ be a morphism and $g, h \in \mathcal{X}_1/y$ such that $f^{-1}(g)$ and $f^{-1}(h)$ exist.*
*(a) If $g \leq h$, then $f^{-1}(g) \leq f^{-1}(h)$.*
*(b) If $g \sim h$, then $f^{-1}(g) \sim f^{-1}(h)$.*

*Proof.* (a) Since $g \leq h$, there exists $\alpha$ such that $g = h \circ \alpha$. Consider the pullback diagrams:

$$\begin{array}{ccc} x_1 & \xrightarrow{t} & y_1 \\ {\scriptstyle f^{-1}(h)}\downarrow & pb & \downarrow{\scriptstyle h} \\ x & \xrightarrow[f]{} & y \end{array} \qquad \begin{array}{ccc} x_2 & \xrightarrow{s} & y_2 \\ {\scriptstyle f^{-1}(g)}\downarrow & pb & \downarrow{\scriptstyle g} \\ x & \xrightarrow[f]{} & y \end{array}$$

So there exists a morphism $k$ such that the triangles in the following diagram commute.




So $f^{-1}(h) \circ k = f^{-1}(g)$, i.e., $f^{-1}(g) \leq f^{-1}(h)$.
(b) Follows from (a). □

**Definition 1.3.** $\mathcal{M}$ has (weak) $\mathcal{X}$−pullbacks if for all $f : x \longrightarrow y$ in $\mathcal{X}$, whenever $(g \sim \mathcal{M}/y)$ $g \in \mathcal{M}/y$, then a pullback, $f^{-1}(g)$, of $g$ along $f$ exists and $(f^{-1}(g) \sim \mathcal{M}/x)$ $f^{-1}(g) \in \mathcal{M}/x$.

**Definition 1.4.** (a) Define $\langle \mathcal{M}/x \rangle = \{\langle n \rangle : n \in \mathcal{M}/x\}$, where $\langle n \rangle = \{n \circ \alpha : \alpha \in \mathcal{X}_1\}$ is the principal sieve generated by $n$, see [5].
(b) Define $[\mathcal{M}/x] = \{[n] : n \in \mathcal{M}/x\}$, where $[n] = \{n \circ \alpha : \alpha \text{ is an iso}\}$.
We regard both $\langle \mathcal{M}/x \rangle$ and $[\mathcal{M}/x]$ as a partially ordered class under inclusion.

**Definition 1.5.** $\mathcal{X}$ is (weakly) $\mathcal{M}$-wellpowered if for all $x$, $(\langle \mathcal{M}/x \rangle)$ $[\mathcal{M}/x]$ is a set.

Let *Mono* denote the collection of all the monomorphisms, then:

**Lemma 1.6.** *(a) If $\mathcal{X}$ is $\mathcal{M}$-wellpowered, then it is weakly $\mathcal{M}$-wellpowered.*
*(b) If $\mathcal{M} \subseteq Mono$, then $\mathcal{X}$ is $\mathcal{M}$-wellpowered if and only if it is weakly $\mathcal{M}$-wellpowered.*

*Proof.* (a) Follows from the fact that the map $\theta : [\mathcal{M}/x] \longrightarrow \langle \mathcal{M}/x \rangle$ defined by $\theta([m]) = \langle m \rangle$ is a surjection.
(b) Follows from the fact that, under the assumption $\mathcal{M} \subseteq Mono$, $\theta$ is a bijection. □

**Remark 1.7.** If $\mathcal{M} \subseteq Mono$, the notion of $\mathcal{M}$-wellpoweredness is equivalent to the one given in [1].

**Definition 1.8.** We say $\mathcal{M}$ is locally complete if for every $x$, $\mathcal{M}/x$ is complete, i.e., it has class-indexed meets.

Note that since $\mathcal{M}/x$ is a preordered class, meets are not unique, not even up to isomorphism. However if we denote by $\mathfrak{M}_A$ the collection of all the meets of $A \subseteq \mathcal{M}/x$, and if $m \in \mathfrak{M}_A$, then $m' \in \mathfrak{M}_A$ if and only if $m' \in \mathcal{M}/x$ and $m' \sim m$.

With $\langle A \rangle = \{\langle m \rangle : m \in A\}$, we have:

**Lemma 1.9.** *$\mathcal{M}/x$ has meets if and only if $\langle \mathcal{M}/x \rangle$ does. In this case for every $A \subseteq \mathcal{M}/x$, $\langle \mathfrak{M}_A \rangle = \mathfrak{M}_{\langle A \rangle}$.*

*Proof.* Suppose $\mathcal{M}/x$ has meets. Given $B \subseteq \langle \mathcal{M}/x \rangle$, let the function $\psi : \langle \mathcal{M}/x \rangle \longrightarrow \mathcal{M}/x$ be a section of the epi $\phi : \mathcal{M}/x \longrightarrow \langle \mathcal{M}/x \rangle$ that takes $m$ to $\langle m \rangle$ and set $A = \psi(B)$. Then $A \subseteq \mathcal{M}/x$ and $B = \langle A \rangle$. Let $m$ be a meet of $A$. Then $\langle m \rangle$ is the meet of $\langle A \rangle$, So $B = \langle A \rangle$ has a meet.

Conversely suppose $\langle \mathcal{M}/x \rangle$ has meets. Given $A \subseteq \mathcal{M}/x$, we have $\langle A \rangle \subseteq \langle \mathcal{M}/x \rangle$. Let $\langle m \rangle$ be the meet of $\langle A \rangle$. Then $m$ is a meet of $A$. Therefore $A$ has a meet.

The last assertion can be easily verified. □

Recall that a map $f : (X, \leq) \longrightarrow (Y, \leq)$ of complete preordered classes is said to be continuous if whenever $m$ is a meet of $A \subseteq X$, $f(m)$ is a meet of $f(A)$.

**Proposition 1.10.** *Let $f : (X, \leq) \longrightarrow (Y, \leq)$ be a map of complete preordered classes. Then the following are equivalent.*

*(a) $f$ is continuous.*

*(b) For all $A \subseteq X$, $f(\mathfrak{M}_A) \cap \mathfrak{M}_{f(A)} \neq \emptyset$.*

*(c) For all $A \subseteq X$, $f(\mathfrak{M}_A) \subseteq \mathfrak{M}_{f(A)}$.*

*Proof.* Straightforward. □

**Definition 1.11.** $\mathcal{M}$ is said to be stably locally complete if it is locally complete, it has weak $\mathcal{X}$-pullbacks, and for every morphism $f : x \longrightarrow y$, $f^{-1} : \overline{\mathcal{M}}/y \longrightarrow \overline{\mathcal{M}}/x$ is continuous, where $\overline{\mathcal{M}} = \{f : f \sim \mathcal{M}\}$.

**Lemma 1.12.** *Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be two collections of morphisms in a category $\mathcal{X}$. If for every $m \in \mathcal{M}$, we have $m \sim \mathcal{M}_0$, then:*

*(a) $\mathcal{M}_0$ has weak $\mathcal{X}$-pullbacks if and only if $\mathcal{M}$ does.*

*(b) $\mathcal{X}$ is weakly $\mathcal{M}_0$-wellpowered if and only if $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered.*

*(c) $\mathcal{M}_0$ is locally complete if and only if $\mathcal{M}$ is.*

*(d) $\mathcal{M}_0$ is stably locally complete if and only if $\mathcal{M}$ is.*

*Proof.* (a) Follows from the fact that $f \sim \mathcal{M}_0$ if and only if $f \sim \mathcal{M}$.
(b) Follows from the fact that $\langle \mathcal{M}_0 \rangle = \langle \mathcal{M} \rangle$.
(c) Follows from Lemma 1.9 and the fact that $\langle \mathcal{M}_0 \rangle = \langle \mathcal{M} \rangle$.
(d) Follows from part (c) and the fact that $\overline{\mathcal{M}_0} = \overline{\mathcal{M}}$. □

## 2. Classes of Morphisms versus Sieve-Presheaves

Let us first introduce the following (quasi)categories that we are going to use in this section.

*Prcls* (*Prset*) denotes the quasicategory (category) of preordered classes (sets) with order preserving maps.

*Cmprcls* (*Cmprset*) denotes the quasicategory (category) of complete preordered classes (sets) with meet preserving maps.

*Pocls* (*Poset*) denotes the quasicategory (category) of partially ordered classes (sets) with order preserving maps.

*Cmpocls* (*Cmposet*) denotes the quasicategory (category) of complete partially ordered classes (sets) with meet preserving maps.

**Definition 2.1.** Each class $\mathcal{M} \subseteq \mathcal{X}_1$ with weak $\mathcal{X}$-pullbacks yields a mapping $M : \mathcal{X}^{op} \longrightarrow Pocls$ defined by:

$$f : x \longrightarrow y \longmapsto f^* : (\langle \mathcal{M}/y \rangle, \subseteq) \longrightarrow (\langle \mathcal{M}/x \rangle, \subseteq)$$

where $f^*(\langle n \rangle) = \langle f^{-1}(n) \rangle$.

We call a functor $M : \mathcal{X}^{op} \longrightarrow Pocls$, an order-presheaf; if in addition $M(x) \subseteq \langle \mathcal{X}_1/x \rangle$ and $M(f) = f^*$, we call it a sieve-presheaf.

In the above definition we assume for all $x \in \mathcal{X}$, $\mathcal{M}/x \neq \emptyset$ (though a weaker condition is needed), which is generally the case for useful collections.

Let $C$ be a subcategory of $B$ with $I : C \hookrightarrow B$ denoting the inclusion. A functor $F : A \longrightarrow B$ is said to factor through $C$ if it factors through the inclusion $I$, i.e., if there is a functor $G : A \longrightarrow C$ such that $F = I \circ G$.

**Proposition 2.2.** *Let $\mathcal{M} \subseteq \mathcal{X}_1$ have weak $\mathcal{X}$-pullbacks and the mapping $M : \mathcal{X}^{op} \longrightarrow Pocls$ correspond to $\mathcal{M}$.*

*(a) $M$ is a sieve-presheaf.*

*(b) $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered if and only if $M$ factors through Poset.*

*(c) $\mathcal{M}$ is stably locally complete if and only if $M$ factors through Cmpocls.*

*(d) $\mathcal{M}$ is stably locally complete and $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered if and only if $M$ factors through Cmposet.*

*Proof.* (a) Given $\xi \in \langle \mathcal{M}/y \rangle$, $\xi = \langle m \rangle$ some $m \in \mathcal{M}/y$. Since $\mathcal{M}$ has weak $\mathcal{X}$-pullbacks, $f^{-1}(m)$ exists and $f^{-1}(m) \sim \mathcal{M}/x$. Therefore $f^*(\xi) = \langle f^{-1}(m) \rangle \in \langle \mathcal{M}/x \rangle$. By Lemma 1.2(a), $f^*$ preserves order. The rest follows easily.

(b) $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered if and only if for all $x \in \mathcal{X}_0$, $\langle \mathcal{M}/x \rangle = M(x)$ is a partially ordered set if and only if $M$ factors through *Poset*.

(c) Suppose $\mathcal{M}$ is stably locally complete. By Lemma 1.9, for each $x$, $M(x)$ is a complete partially ordered class. Now let $B \subseteq \langle \mathcal{M}/y \rangle$. $B = \langle A \rangle$ for a class $A \subseteq \mathcal{M}/y$ and we have $f^*(\mathfrak{M}_B) = f^*(\mathfrak{M}_{\langle A \rangle}) = f^*(\langle \mathfrak{M}_A \rangle) = \langle f^{-1}(\mathfrak{M}_A) \rangle$, where $\mathfrak{M}_A$ is the collection of meets of $A$ in $\overline{\mathcal{M}}/y$. Since $f^{-1}$ is continuous, $\langle f^{-1}(\mathfrak{M}_A) \rangle \subseteq \langle \mathfrak{M}_{f^{-1}(A)} \rangle$, where $\mathfrak{M}_{f^{-1}(A)}$ is the collection of meets of $f^{-1}(A)$ in $\overline{\mathcal{M}}/x$. So $f^*(\mathfrak{M}_B) \subseteq \langle \mathfrak{M}_{f^{-1}(A)} \rangle = \mathfrak{M}_{\langle f^{-1}(A) \rangle} = \mathfrak{M}_{f^*(\langle A \rangle)} = \mathfrak{M}_{f^*(B)}$, rendering $f^*$ continuous.

Conversely suppose $M$ factors through *Cmpocls*. By Lemma 1.9, for each $x$, $\mathcal{M}/x$ is a complete partially ordered class. Hence $\mathcal{M}$ is locally complete. Now let $f : x \longrightarrow y$ and $A \subseteq \overline{\mathcal{M}}/y$. If $n \in f^{-1}(\mathfrak{M}_A)$, then $n = f^{-1}(m)$, with $m \in \mathfrak{M}_A$. Therefore $\langle m \rangle \in \langle \mathfrak{M}_A \rangle = \mathfrak{M}_{\langle A \rangle}$ and so $f^*(\langle m \rangle) \in f^*(\mathfrak{M}_{\langle A \rangle})$. Since $f^*$ is continuous, $f^*(\mathfrak{M}_{\langle A \rangle}) \subseteq \mathfrak{M}_{f^*(\langle A \rangle)}$. So $f^*(\langle m \rangle) \in \mathfrak{M}_{f^*(\langle A \rangle)} = \mathfrak{M}_{\langle f^{-1}(A) \rangle} = \langle \mathfrak{M}_{f^{-1}(A)} \rangle$. Therefore $\langle f^{-1}(m) \rangle = f^*(\langle m \rangle) \in \langle \mathfrak{M}_{f^{-1}(A)} \rangle$ and so $\langle f^{-1}(m) \rangle = \langle j \rangle$ with $j \in \mathfrak{M}_{f^{-1}(A)}$. It follows that $f^{-1}(m) \sim j$ with $j \in \mathfrak{M}_{f^{-1}(A)}$. Therefore $n = f^{-1}(m) \in \mathfrak{M}_{f^{-1}(A)}$. Hence $f^{-1}(\mathfrak{M}_A) \subseteq \mathfrak{M}_{f^{-1}(A)}$ and by Proposition 1.10, $f^{-1}$ is continuous.

(d) Follows from (b) and (c). □

**Definition 2.3.** Each sieve-presheaf $M$ yields a class $\mathcal{M} \subseteq \mathcal{X}_1$ defined by:

$$\mathcal{M} = \cup_{x \in \mathcal{X}_0} \{ m : \langle m \rangle \in M(x) \}$$

**Definition 2.4.** A class $\mathcal{M} \subseteq \mathcal{X}_1$ is said to be saturated if $f \sim \mathcal{M}$ implies $f \in \mathcal{M}$.

**Remark 2.5.** Note that $\mathcal{M}$ is saturated if and only if $\overline{\mathcal{M}} = \mathcal{M}$ and that a saturated class $\mathcal{M}$ has weak $\mathcal{X}$-pullbacks if and only if it has $\mathcal{X}$-pullbacks.

**Proposition 2.6.** *Let $M : \mathcal{X}^{op} \longrightarrow Pocls$ be a sieve-presheaf and $\mathcal{M}$ be its associated class of morphisms.*

*(a) $\mathcal{M}$ is saturated and has $\mathcal{X}$-pullbacks.*

*(b) $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered if and only if $M$ factors through Poset.*
*(c) $\mathcal{M}$ is stably locally complete if and only if $M$ factors through Cmpocls.*
*(d) $\mathcal{M}$ is stably locally complete and $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered if and only if $M$ factors through Cmposet*

*Proof.* (a) Let $f \sim \mathcal{M}$. Then there is $m \in \mathcal{M}$ such that $f \sim m$. So $\langle f \rangle = \langle m \rangle \in M(x)$, where $x$ is the codomain of $f$. Hence $f \in \mathcal{M}$. Therefore $\mathcal{M}$ is saturated.

Let $x \xrightarrow{f} y$ be a morphism. Since $M$ is a sieve-presheaf, $M(f) = f^*$. Now if $m \in \mathcal{M}/y$, then $\langle m \rangle \in M(y)$. So $\langle f^{-1}(m) \rangle = f^*(\langle m \rangle) = M(f)(\langle m \rangle) \in M(x)$. Therefore $f^{-1}(m) \in \mathcal{M}$. Hence $\mathcal{M}$ has $\mathcal{X}$-pullbacks.

(b) and (c) follow easily by observing that for each $x \in \mathcal{X}$, $\langle \overline{\mathcal{M}}/x \rangle = \langle \mathcal{M}/x \rangle = M(x)$.

(d) Follows from (b) and (c). □

**Theorem 2.7.** *There is a one to one correspondence between:*
*(a) saturated subclasses of $\mathcal{X}_1$ that have $\mathcal{X}$-pullbacks and sieve-presheaves.*
*(b) saturated subclasses of $\mathcal{X}_1$ that have $\mathcal{X}$-pullbacks such that $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered and sieve-presheaves that factor through Poset.*
*(c) saturated subclasses of $\mathcal{X}_1$ that have $\mathcal{X}$-pullbacks and are stably locally complete and sieve-presheaves that factor through Cmpocls.*
*(d) saturated subclasses of $\mathcal{X}_1$ that have $\mathcal{X}$-pullbacks and are stably locally complete such that $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered and sieve-presheaves that factor through Cmposet.*

*Proof.* (a) Given $\mathcal{M} \subseteq \mathcal{X}_1$ a saturated class with $\mathcal{X}$-pullbacks, let $M$ be the corresponding sieve-presheaf, see Proposition 2.2 (a) and $\mathcal{M}'$ be its corresponding class of morphisms, see Proposition 2.6 (a). It is always the case that $\mathcal{M} \subseteq \mathcal{M}'$. Since $\mathcal{M}$ is saturated, it follows that $\mathcal{M}' \subseteq \mathcal{M}$, so that $\mathcal{M} = \mathcal{M}'$.

Now given a sieve-presheaf $M$, one can easily verify that the subobject $M'$ corresponding to the class $\mathcal{M} = \cup_{x \in \mathcal{X}_0} \{m : \langle m \rangle \in M(x)\} \subseteq \mathcal{X}_1$ equals $M$.

(b), (c) and (d) follow from part (a) and respectively parts (b), (c) and (d) of Propositions 2.2 and 2.6. □

## 3. Examples

**Lemma 3.1.** *In Set, $f \leq g$ if and only if $I_f \subseteq I_g$, where $I_f$ denotes the image of $f$; and so $f \sim g$ if and only if $I_f = I_g$.*

*Proof.* Straightforward. □

**Lemma 3.2.** *Let $U : \mathcal{X} \longrightarrow Set$ be a functor and $\mathcal{M} \subseteq \mathcal{X}_1$. For each $x \in \mathcal{X}$, define $\phi_x : \langle \mathcal{M}/x \rangle \longrightarrow P(U(x))$ by $\phi_x(\langle m \rangle) = I_{U(m)}$, where $I_{U(m)} \subseteq U(x)$ is the image of $U(m)$. If $\langle m \rangle = \langle n \rangle$ ($n$ need not be in $\mathcal{M}/x$), then $m \sim n$ and so $U(m) \sim U(n)$. It follows that $I_{U(m)} = I_{U(n)}$. Hence $\phi_x$ is well-defined. Denoting by $Ini$ ($IniMono$) the collection of all $U$-initial morphisms ($U$-initial monomorphisms), we have:*

*(a) If $\mathcal{M} \subseteq Ini$ ($\mathcal{M} \subseteq IniMono$), then $\phi_x$ is one to one and so $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered ($\mathcal{M}$-wellpowered).*

*(b) If the structured inclusions have initial $\mathcal{M}$-lifts, then $\phi_x$ is onto.*

*(c) If $\mathcal{M} \subseteq Ini$ and structured inclusions have $\mathcal{M}$-lifts, then $\phi_x$ is an isomorphism in Poset, $\langle \mathcal{M}/x \rangle$ is complete (so $\mathcal{M}$ is locally complete) and $\phi_x$ is indeed an isomorphism in Cmposet.*

*(d) If $\mathcal{M}$ has weak $\mathcal{X}$-pullbacks, is locally complete and for each $x \in \mathcal{X}$, $\overline{\mathcal{M}}/x$-meets are $\mathcal{X}_1/x$-meets, then $\mathcal{M}$ is stably locally complete.*

*(e) If $\mathcal{M} = Ini$, $\mathcal{X}$ has pullbacks and $U$ preserves them, then $\mathcal{M}$ has $\mathcal{X}$-pullbacks.*

*(f) If $\mathcal{M} = IniMono$ and $\mathcal{X}$ has pullbacks, then $\mathcal{M}$ has $\mathcal{X}$-pullbacks.*

*(g) If $\mathcal{M}$ has weak $\mathcal{X}$-pullbacks, then for any $f \in \mathcal{X}_1/x$ and any $m \in \mathcal{M}/x$, $I_{U(f^{-1}(m))} \leq U(f)^{-1}(I_{U(m)})$. If furthermore $U$ preserves such pullbacks, then we get $I_{U(f^{-1}(m))} = U(f)^{-1}(I_{U(m)})$.*

*(h) If $\mathcal{M} \subseteq Ini$ has weak $\mathcal{X}$-pullbacks, $U$ preserves such pullbacks and structured inclusions have $\mathcal{M}$-lifts, then $\mathcal{M}$ is stably locally complete.*

*Proof.* (a) Suppose $\phi(\langle m \rangle) = \phi(\langle n \rangle)$. Then $I_{U(m)} = I_{U(n)}$ and so by Lemma 3.1, $U(m) \sim U(n)$. Initiality of $m$ and $n$ implies $m \sim n$ and so $\langle m \rangle = \langle n \rangle$. It follows that $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered. In case $\mathcal{M} \subseteq IniMono$, $\mathcal{M}$-wellpoweredness follows from Lemma 1.6 (b).

(b) Given $A \subseteq U(x)$, let $m : a \longrightarrow x$ in $\mathcal{M}$ be an initial lift of the structured inclusion $A \subseteq U(x)$. Then $\phi(\langle m \rangle) = A$.

(c) By (a) and (b) $\phi_x$ is an isomorphism in *Set*. One can easily verify that $\phi_x$ preserves order. Now if $A \subseteq B \subseteq U(x)$, then $\phi_x^{-1}$ takes $A$ and $B$ to $\langle m \rangle$

and $\langle n \rangle$, where $m$ and $n$ are initial lifts of $A \subseteq U(x)$ and $B \subseteq U(x)$. It then follows that the inclusion $A \subseteq B$ has a lift to say $j$ such that $m = nj$ and so $m \leq n$ or equivalently $\langle m \rangle \subseteq \langle n \rangle$. This makes $\phi_x$ an isomorphism in *Poset*. Now the fact that $P(U(x))$ is complete, makes $\langle \mathcal{M}/x \rangle$ complete and so by Lemma 1.9, $\mathcal{M}/x$ is complete. Finally it is obvious that $\phi_x$ preserves meets; using this, it then follows that so does $\phi_x^{-1}$. Hence $\phi_x$ is an isomorphism in *Cmposet*.

(d) To show stability, let $m_i : a_i \longrightarrow x$ be in $\mathcal{M}/x$, for $i \in I$, $m : a \longrightarrow x$ be a meet of $m_i$ and $f : y \longrightarrow x$ be a morphism. Form the pullbacks:

$$\begin{array}{ccc} b_i & \xrightarrow{f_i^*} & a_i \\ {\scriptstyle n_i}\downarrow & p.b. & \downarrow{\scriptstyle m_i} \\ y & \xrightarrow[f]{} & x \end{array} \quad \text{and} \quad \begin{array}{ccc} a^* & \xrightarrow{f^*} & a \\ {\scriptstyle m^*}\downarrow & p.b. & \downarrow{\scriptstyle m} \\ y & \xrightarrow[f]{} & x \end{array}$$

and let $n : b \longrightarrow y$ be a meet of $n_i$. We need to show $m^* \sim n$. We have morphisms $f_i : a \longrightarrow a_i$ such that $m_i \circ f_i = m$ and $g_i : b \longrightarrow b_i$ such that $n_i \circ g_i = n$. Since $m_i \circ f_i \circ f^* = f \circ m^*$ , the first pullback yields $k_i : a^* \longrightarrow b_i$ such that $n_i \circ k_i = m^*$ and $f_i^* \circ k_i = f_i \circ f^*$. It follows that $m^* \leq n_i$ and so $m^* \leq n$. On the other hand, since $f \circ n = m_i \circ (f_i^* \circ g_i)$, $f \circ n \leq m_i$. Since $\overline{\mathcal{M}}/x$-meets are $\mathcal{X}_1/x$-meets and $f \circ n \in \mathcal{X}_1/x$, $f \circ n \leq m$. So there is $l : b \longrightarrow a$ such that $f \circ n = m \circ l$. Now the second pullback yields a morphism $t : b \longrightarrow a^*$ such that $f^* \circ t = l$ and $m^* \circ t = n$. It follows that $n \leq m^*$. Thus $m^* \sim n$, as desired.

(e) Let $m : b \longrightarrow y$ be initial and $f : x \longrightarrow y$ be a morphism. Consider the pullback:

$$\begin{array}{ccc} a & \xrightarrow{f^*} & b \\ {\scriptstyle m^*}\downarrow & p.b. & \downarrow{\scriptstyle m} \\ x & \xrightarrow[f]{} & y \end{array}$$

Given $c \in \mathcal{X}$, $g : U(c) \longrightarrow U(a)$ and $\bar{h} : c \longrightarrow x$ a lift of $U(m^*) \circ g$, $f \circ \bar{h}$ is a lift of $U(m) \circ (U(f^*) \circ g)$. Since $m$ is initial, there is a lift $\bar{k} : c \longrightarrow b$ of $U(f^*) \circ g$ such that $f \circ \bar{h} = m \circ \bar{k}$. Now the above pullback yields a morphism $\bar{g} : c \longrightarrow a$ such that $m^* \circ \bar{g} = \bar{h}$ and $f^* \circ \bar{g} = \bar{k}$. Since $U$ preserves pullback,

applying $U$ to the above pullback, we get a pullback diagram in $Set$. Using this pullback, we get $U(\bar{g}) = g$ and so $\bar{g}$ is the required lift of $g$.

(f) We know the pullback of a mono is a mono, see [1]. This and a proof similar to (e) shows pullback of an initial morphism is initial without the assumption that $U$ preserves pullbacks.

(g) Let $f : y \longrightarrow x$ be a morphism and $m : a \longrightarrow x$ be in $\mathcal{M}/x$. The images $I_{U(m)}$ and $I_{U(f^{-1}(m))}$ are obtained as the following diagrams show.

$$\begin{array}{ccc} U(a) & \xrightarrow{U(m)} & U(x) \\ & \overset{e}{\searrow} \quad /// \quad \overset{i}{\nearrow} & \\ & I_{U(m)} & \end{array} \quad \text{and} \quad \begin{array}{ccc} U(b) & \xrightarrow{U(f^{-1}(m))} & U(y) \\ & \overset{e'}{\searrow} \quad /// \quad \overset{i'}{\nearrow} & \\ & I_{U(f^{-1}(m))} & \end{array}$$

We have the following pullbacks.

$$\begin{array}{ccc} b & \xrightarrow{f^*} & a \\ {\scriptstyle f^{-1}(m)}\downarrow & p.b. & \downarrow{\scriptstyle m} \\ y & \xrightarrow[f]{} & x \end{array} \quad \text{and} \quad \begin{array}{ccc} U(f)^{-1}(I_{U(m)}) & \xrightarrow{g} & I_{U(m)} \\ {\scriptstyle i^*}\downarrow & p.b. & \downarrow{\scriptstyle i} \\ U(y) & \xrightarrow[U(f)]{} & U(x) \end{array}$$

Let $s'$ be the right inverse $e'$ and use the maps $i'$ and $eU(f^*)s'$ in the above right pullback to get $i' \leq i^*$ an so $I_{U(f^{-1}(m))} \subseteq U(f)^{-1}(I_{U(m)})$.

Under the additional hypothesis, we have the following pullback diagram.

$$\begin{array}{ccc} U(b) & \xrightarrow{U(f^*)} & U(a) \\ {\scriptstyle U(f^{-1}(m))}\downarrow & p.b. & \downarrow{\scriptstyle U(m)} \\ U(y) & \xrightarrow[U(f)]{} & U(x) \end{array}$$

Let $s$ be the right inverse of $e$ and use the maps $i^*$ and $sg$ in the above pullback to get $i^* \leq U(f^{-1}(m))$ and so $U(f)^{-1}(I_{U(m)}) \subseteq I_{U(f^{-1}(m))}$.

(h) Given $f : y \longrightarrow x$ in $\mathcal{X}$, by part (g) we have the following commutative diagram.

$$\begin{array}{ccc} \langle \mathcal{M}/x \rangle & \xrightarrow{\phi_x} & P(U(x)) \\ {\scriptstyle f^*}\downarrow & /// & \downarrow{\scriptstyle U(f)^{-1}} \\ \langle \mathcal{M}/y \rangle & \xrightarrow[\phi_y]{} & P(U(y)) \end{array}$$

Since by part (c), $\mathcal{M}/x$ and $\mathcal{M}/y$ are complete and $\phi_x$ and $\phi_y$ are isomorphisms in *Cmposet* and since $U(f)^{-1}$ is continuous, so is $f^*$. Using the fact that $\langle \mathcal{M}/x \rangle = \langle \overline{\mathcal{M}}/x \rangle$, Lemma 1.9 implies $f^{-1} : \overline{\mathcal{M}}/x \longrightarrow \overline{\mathcal{M}}/y$ is continuous.

□

**Example 3.3.** *(a) Let $\mathcal{X} = Set$ and $U = 1 : Set \longrightarrow Set$.*

*(i) Let $\mathcal{M} = \mathcal{X}_1$. Then obviously $\mathcal{M} = Ini$ and so Lemma 3.2 parts (a), (e) and (h) yields $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered, $\mathcal{M}$ has $\mathcal{X}$-pullbacks and is stably locally complete. Note $\mathcal{X}$ is not $\mathcal{M}$-wellpowered.*

*(ii) Let $\mathcal{M}_0 = \{m : m \text{ is an inclusion}\}$ and $\mathcal{M}$ be as in (i). For each $f : X \longrightarrow Y$ in $\mathcal{M}$, let $f = me$ be the (epi,mono)-factorization of $f$, with $m : I_f \longrightarrow Y$ the inclusion of the image of $f$ to $Y$. $f = me$ implies $f \leq m$ and since $e$ is a retraction, there is $s$ such that $es = 1$. It follows that $fs = m$ and so $m \leq f$. Hence $f \sim \mathcal{M}_0$. By Lemma 1.12, $\mathcal{X}$ is weakly $\mathcal{M}_0$-wellpowered, $\mathcal{M}_0$ has weak $\mathcal{X}$-pullbacks and is stably locally complete. Since a pullback of an inclusion can be chosen to be the inclusion, $\mathcal{M}_0$ has in fact $\mathcal{X}$-pullbacks. By Lemma 1.6 (b), $\mathcal{X}$ is indeed $\mathcal{M}_0$-wellpowered.*

*(iii) Let $\mathcal{M}_0 = \{m : m \text{ is an inclusion}\}$ and $\mathcal{M}$ be any collection of morphisms containing $\mathcal{M}_0$ (for example $\mathcal{M} = Mono$). Then for each $f : X \longrightarrow Y$ in $\mathcal{M}$ we have $f \sim m$. So by Lemma 1.12, $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered, $\mathcal{M}$ has weak $\mathcal{X}$-pullbacks and is stably locally complete.*

*(b) Let $\mathcal{X} = Top$ and $U : Top \longrightarrow Set$ be the forgetful functor.*

*(i) Let $\mathcal{M} = \{f : (X, \tau_X) \longrightarrow (Y, \tau_Y) : \tau_X \text{ is the initial topology}\}$. Then $\mathcal{M} = Ini$, see* [1]. *So by Lemma 3.2 parts (a), (e) and (h), $\mathcal{X}$ is weakly $\mathcal{M}$-wellpowered, $\mathcal{M}$ has $\mathcal{X}$-pullbacks and is stably locally complete.*

*(ii) Let $\mathcal{M}_0 = IniMono$ and $\mathcal{M}$ be as in (i). Since in the (epi,initial mono)-factorization of an initial $f$, the epi part is a retraction, $f \sim \mathcal{M}_0$. So by Lemma 1.12, $\mathcal{X}$ is weakly $\mathcal{M}_0$-wellpowered, $\mathcal{M}_0$ has weak $\mathcal{X}$-pullbacks and is stably locally complete. Since a pullback of an initial mono is an initial mono, $\mathcal{M}_0$ has $\mathcal{X}$-pullbacks; and since $\mathcal{M}_0 \subseteq Mono$, by Lemma 1.6 (b), $\mathcal{X}$ is $\mathcal{M}_0$-wellpowered.*

*(iii) Let $\mathcal{M} = Mono$. By Remark 1.7 and* [1], *$\mathcal{X}$ is $\mathcal{M}$-wellpowerd, in fact for $X \in Top$, $\langle \mathcal{M}/X \rangle = [\mathcal{M}/X]$ is in bijection with the set of all the lifts of the structured inclusions to $U(X)$. Obviously $\mathcal{M}$ has $\mathcal{X}$-pullbacks.*

*Since intersections in Top exist, see* [1], *and are monos, meets coinside with intersections; hence* $\mathcal{M}$ *is locally complete. In fact* $\mathcal{M}$ *is stably locally complete, because pullbacks preserve intersections.*

*(iv) For a continuous function* $m : (A, \tau_A) \longrightarrow (X, \tau_X)$, *let the function* $m^c : (A, \tau_A) \longrightarrow (X, \tau_X^c)$ *have the same underlying function as* $m$, *where* $\tau_X^c$ *is the coinduced topology by* $m$ *and* $\tau_A$. *Let* $\mathcal{M} = \{m \in Epi : m^c$ *is a retraction* $\}$. *Each* $1_X : (X, \tau_X') \longrightarrow (X, \tau_X)$ *belongs to* $\mathcal{M}$ *and each* $m : (A, \tau_A) \longrightarrow (X, \tau_X)$ *is related to some* $1_X : (X, \tau_X^c) \longrightarrow (X, \tau_X)$. *So* $\mathcal{X}$ *is weakly* $\mathcal{M}$*-wellpowered (but obviously not* $\mathcal{M}$*-wellpowered). Since pullback of each* $1_X$ *is a* $1_Y$ *and each* $m \in \mathcal{M}$ *is related to a* $1_X$, $\mathcal{M}$ *has weak* $\mathcal{X}$*-pullbacks. Let* $m_i : (A_i, \tau_i) \longrightarrow (X, \tau_X)$ *be in* $\mathcal{M}/(X, \tau_X)$, *where* $i$ *is in any class* $I$. *Each* $m_i \sim 1_i$, *where* $1_i : (X, \tau_i) \longrightarrow (X, \tau_X)$. *Since the collection* $\{1_i : i \in I\}$ *has a meet* $1_X : (X, \tau_X') \longrightarrow (X, \tau_X)$, *where* $\tau_X'$ *is the topology generated by* $\cup\tau_i$, *so does the collection* $\{m_i : i \in I\}$. *Hence* $\mathcal{M}$ *is locally complete. One can verify* $\mathcal{M}$ *is indeed stably locally complete.*

*(c) Let* $\mathcal{X} = Rel$ *and* $U : Rel \longrightarrow Set$ *be the forgetful functor, see* [1].

*(i) Let* $\mathcal{M} = \{f : (X, R_X) \longrightarrow (Y, R_Y) : xR_Xy$ *iff* $f(x)R_Yf(y)\}$. *Then* $\mathcal{M}$ *is the collection of initial morphisms, see* [1]. *Similar to part (b) (i),* $\mathcal{X}$ *is weakly* $\mathcal{M}$*-wellpowered,* $\mathcal{M}$ *has* $\mathcal{X}$*-pullbacks and is stably locally complete.*

*(ii)* $\mathcal{M}_0 = IniMono$. *Similar to part b (ii),* $\mathcal{X}$ *is* $\mathcal{M}_0$*-wellpowered,* $\mathcal{M}_0$ *has* $\mathcal{X}$*-pullbacks and is stably locally complete.*

*(iii)* $\mathcal{M} = Mono$. *Similar to part b (iii),* $\mathcal{X}$ *is* $\mathcal{M}$*-wellpowered,* $\mathcal{M}$ *has* $\mathcal{X}$*-pullbacks and is stably locally complete.*

*(d) Let* $\mathcal{X} = Haus$ *and* $U : Haus \longrightarrow Set$ *be the forgetful functor, see* [1].

*(i) Let* $\mathcal{M}$ *be the collection of topological embeddings. Then* $\mathcal{M} = Ini$, *see* [1] *and similar to part (b) (i),* $\mathcal{X}$ *is weakly* $\mathcal{M}$*-wellpowered,* $\mathcal{M}$ *has* $\mathcal{X}$*-pullbacks and is stably locally complete.*

*(ii) Let* $\mathcal{M} = Epi$ *be the collection of epis. Since epis in Haus are not preserved by pullbacks, see* [1] *and since* $\overline{Epi} = Epi$, $\mathcal{M}$ *does not have weak* $\mathcal{X}$*-pullbacks. Also* $\mathcal{M}$ *is not locally complete, as the epis* $i : Q \hookrightarrow R$ *from rationals to reals and* $j : R - Q \hookrightarrow R$ *do not have a meet (any* $h$ *less that*

*or equal to both, has image contained in* $Q \cap (R - Q) = \emptyset$. *and so* $h$ *is the inclusion from the empty set and is therefore not in Epi.*

**Example 3.4.** *In the following,* $M : \mathcal{X}^{op} \longrightarrow Pocls$ *is the sieve-presheaf associated to the given collection* $\mathcal{M}$, *and* $P : Set^{op} \longrightarrow Cmposet$ *is the power set functor. Using Example 3.3 and Proposition 2.2, we have:*

*(a) For* $\mathcal{X} = Set$ *and* $\mathcal{M}$ *any collection of morphisms containing all the inclusions,* $M$ *factors through Cmposet and is naturally isomorphic to* $P \circ U^{op}$ *in* $Cmposet^{\mathcal{X}^{op}}$.

*(b) For* $\mathcal{X} = Top$ *and* $\mathcal{M} = Ini$ *or* $\mathcal{M} = IniMono$, $M$ *factors through Cmposet and is naturally isomorphic to* $P \circ U^{op}$ *in* $Cmposet^{\mathcal{X}^{op}}$.

*For* $\mathcal{M} = Mono$, $M$ *factors through Cmposet and is naturally isomorphic to the functor that takes* $X$ *to the set of all the lifts of the structured inclusions to* $U(X)$.

*For* $\mathcal{M} = \{m \in Epi : m^c$ *is a retraction* $\}$, $M$ *factors through Cmposet and is naturally isomorphic to the functor that takes* $X$ *to the set of all the topologies on* $X$ *containing the given topology on* $X$.

*(c) For* $\mathcal{X} = Rel$ *and* $\mathcal{M} = Ini$, $\mathcal{M} = IniMono$, *or* $\mathcal{M} = Mono$, *we have results similar to part (b).*

*(d) For* $\mathcal{X} = Haus$ *and* $\mathcal{M} = Ini$, $M$ *factors through Cmposet. However for* $\mathcal{M} = Epi$, *since* $\mathcal{M}$ *does not have* $\mathcal{X}$*-pullbacks, the mapping* $M$ *is not defined.*

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# THE CLASSIFICATION OF LOW-DIMENSIONAL 2-LIE SUPERALGEBRAS

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## Abstract

In this paper, we give the classification of 2-Lie superalgebras with dimension no more than four over a field F of characteristic 0. We classify 2-Lie superalgebras into fifty classes by the dimension of $< L, L >$ and the Jacobi identity. When the dimension of $L$ is one, it has one class; when the dimension of $L$ is two, it has four classes; when the dimension of $L$ is three, it has fifteen classes; when the dimension of $L$ is four, it has thirty classes.
**Key words:** 2-Lie superalgebras, Classification, Jacobi identity.
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## §1 Introduction

Lie superalgebras arose initially in the 1970s, which play an important role in physics and have a close relationship with Lie algebras. The classification of finite dimensional complex semisimple Lie algebra and the classification of real semisimple Lie algebra had been completely resolved [1].

In 1970, Kac made a comprehensive exposition of Lie superalgebras, and obtained the complete classification of the simple (finite-dimensional) Lie superalgebras over an algebraically closed field of characteristic zero [2]. Recently, parts of the results of the classification of Lie superalgebras of low dimensions have been developed by Z. G. Wang [3]. Wang classified the Lie superalgebras with dimension $\leq 4$ over an algebraically closed field of characteristic zero. In [4], Zhang determined the types of Lie superalgebras of dimension two and three over any field F of charF $\neq 2$ by the method of undetermined coefficient. In [5], Ni continued the research of Zhang and determined the type of Lie superalgebras of dimension four. In [6], Nicoletta Cantarini and Kac clssified simple linearly compact $n$-Lie superalgebras with $n > 2$ over a field F of characteristic 0. It is the main object of this paper to determine the types of low-dimensional 2-Lie superalgebras over any field F of characteristic 0.

## §2 Preliminaries

**Definition 2.1.**[2] Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a superalgebra whose multiplication is denoted by $<,>$. $L$ is called as a Lie superalgebra if the multiplication satisfies the following identities:

$$< x, y >= -(-1)^{\alpha\beta} < y, x >,$$

$$(-1)^{\gamma\alpha} < x, < y, z >> +(-1)^{\alpha\beta} < y, < z, x >> +(-1)^{\beta\gamma} < z, < x, y >>= 0,$$

for all $x \in L_\alpha$,$y \in L_\beta$,$z \in L_\gamma$; $\alpha, \beta, \gamma \in \mathrm{Z}_2$.

**Definition 2.2.**[6] Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a superalgebra whose multiplication is denoted by $<,>$. $L$ is called as a 2-Lie superalgebra if the multiplication satisfies the following identities:

$$< x, y >= -(-1)^{\alpha\beta} < y, x >,$$

$$< x, < y, z >>= (-1)^{\theta\alpha}(<< x, y >, z > +(-1)^{\alpha\beta} < y, < x, z >>),$$

for all $x \in L_\alpha$, $y \in L_\beta$, $z \in L_\gamma$; $\alpha, \beta, \gamma \in Z_2$, $\theta \in \{0, 1\}$ is fixed. In this paper, only the case "$\theta = 1$" is considered.

The following four Jacobi identities will be used throughout this paper:

$$(1) < y_1, < y_2, y_3 >> - < y_2, < y_1, y_3 >> + < y_3, < y_1, y_2 >>= 0;$$

$$(2) < x_1, < y_1, y_2 >> - < y_1, < x_1, y_2 >> + < y_2, < x_1, y_1 >>= 0;$$

$$(3) < x_1, < x_2, y_1 >> - < x_2, < x_1, y_1 >> - < y_1, < x_1, x_2 >>= 0;$$

$$(4) < x_1, < x_2, x_3 >> - < x_2, < x_1, x_3 >> + < x_3, < x_1, x_2 >>= 0,$$

for all $x_i \in L_{\bar{0}}, y_i \in L_{\bar{1}}$.

## §3 Main Results

**Theorem** Let $L$ be a 2-Lie superalgebra with dimension no more than four over a field F of characteristic 0. Then $L$ is one of following types up to isomorphism:

(1) $\dim L = 1$

$L_1^1 :< L, L >= 0.$

(2) $\dim L = 2$

$L_2^1 :< L, L >= 0.$

$L_2^2 :< x, y >= x, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0.$

$L_2^3 :< y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_2, y_2 >= y_1.$

$L_2^4 :< y_1, y_1 >= 0, < y_1, y_2 >= y_1, < y_2, y_2 >= 0.$

(3) $\dim L = 3$

$L_3^1 :< L, L >= 0.$

$L_3^2 :< x, y_1 >= x, < x, y_2 >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0.$

$L_3^3 :< y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_2, y_2 >= y_1, < L_{\bar{0}}, L_{\bar{0}} >= 0,$
$< L_{\bar{0}}, L_{\bar{1}} >= 0.$

$L_3^4 :< y_1, y_1 >= 0, < y_1, y_2 >= y_1, < y_2, y_2 >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0,$
$< L_{\bar{0}}, L_{\bar{1}} >= 0.$

$L_3^5 :< y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_2, y_2 >= ky_1, < x, y_1 >= 0,$
$< x, y_2 >= x, < L_{\bar{0}}, L_{\bar{0}} >= 0.$

$L_3^6 :< y_1, y_1 >= 0, < y_1, y_2 >= k_1 y_1, < y_2, y_2 >= 0, < x, y_1 >= 0,$
$< x, y_2 >= x, < L_{\bar{0}}, L_{\bar{0}} >= 0.$

$L_3^7 :< x_1, y >= x_1, < x_2, y >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0,$
$< L_{\bar{1}}, L_{\bar{1}} >= 0.$

$L_3^8 :< x_1, x_1 >= 0, < x_1, x_2 >= y, < x_2, x_2 >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0,$
$< L_{\bar{0}}, L_{\bar{1}} >= 0.$

$L_3^9 :< x_1, x_1 >= 0, < x_1, x_2 >= y, < x_2, x_2 >= 0, < x_1, y >= 0,$
$< x_2, y >= k_4 x_1, < L_{\bar{1}}, L_{\bar{1}} >= 0.$

$L_3^{10} :< x_1, y >= b_{11}x_1 + b_{12}x_2, < x_2, y >= b_{21}x_1 + b_{22}x_2,$
$< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0,$

$$\operatorname{rank} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = 2.$$

$L_3^{11} :< x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y, < x_2, x_2 >= a_{22}y,$
$< x_1, y >= b_{11}x_1 + b_{12}x_2, < x_2, y >= b_{21}x_1 + b_{22}x_2, < L_{\bar{0}}, L_{\bar{0}} >= 0,$
$< L_{\bar{1}}, L_{\bar{1}} >= 0, a_{12} \neq 0,$

$$\operatorname{rank} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{pmatrix} = 2.$$

$L_3^{12} :< y_1, y_1 >= 0, < y_1, y_2 >= b_{12}y_1, < y_1, y_3 >= b_{13}y_1, < y_2, y_2 >= 0,$
$< y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= 0,\ b_{12}^2 + b_{13}^2 + b_{23}^2 \neq 0.$

$L_3^{13} :< y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_1, y_3 >= 0, < y_2, y_2 >= b_{22}y_1,$
$< y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= b_{33}y_1, b_{22}^2 + b_{23}^2 + b_{33}^2 \neq 0.$

$L_3^{14} :< y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2,$
$< y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2, < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2, < y_2, y_3 >= a_{23}^1 y_1 +$
$a_{23}^2 y_2, < y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2,$

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2 \end{pmatrix} = 2.$$

$L_3^{15} :< y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2 + a_{11}^3 y_3, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2 + a_{12}^3 y_3,$
$< y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2 + a_{13}^3 y_3, < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2 + a_{22}^3 y_3,$
$< y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2 + a_{23}^3 y_3,$
$< y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2 + a_{33}^3 y_3,$

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2 \\ a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{22}^3 & a_{23}^3 & a_{33}^3 \end{pmatrix} = 3.$$

(4) $\dim L = 4$

$L_4^1 :< L, L >= 0.$

$L_4^2 :< x_1, y >= a_1 x_1, < x_2, y >= x_1, < x_3, y >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0,$
$< L_{\bar{1}}, L_{\bar{1}} >= 0.$

$L_4^3 :< L_{\bar{1}}, L_{\bar{1}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0, < x_1, x_1 >= 0, < x_1, x_2 >= a_{12} y,$
$< x_1, x_3 >= a_{13} y, < x_2, x_2 >= 0, < x_2, x_3 >= a_{23} y, < x_3, x_3 >= 0, a_{12}^2 +$
$a_{13}^2 + a_{23}^2 \neq 0.$

$L_4^4 :< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, < x_1, y >= a_{11} x_1 + a_{12} x_2, < x_2, y >=$
$a_{21} x_1 + a_{22} x_2, < x_3, y >= a_{31} x_1 + a_{32} x_2,$

$$\operatorname{rank} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} = 2.$$

$L_4^5 :< x_1, y >= 0, < x_2, y >= b_2 x_1, < x_3, y >= b_3 x_1, < x_1, x_1 >= 0,$
$< x_1, x_2 >= a_{12} y, < x_1, x_3 >= a_{13} y, < x_2, x_2 >= 0, < x_2, x_3 >= a_{23} y,$
$< x_3, x_3 >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0; a_{12}^2 + a_{13}^2 + a_{23}^2 \neq 0, b_2^2 + b_3^2 \neq 0.$

$L_4^6 :< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, < x_1, y >= a_{11}x_1 + a_{12}x_2 + a_{13}x_3,$
$< x_2, y >= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, < x_3, y >= a_{31}x_1 + a_{32}x_2 + a_{33}x_3,$

$$\operatorname{rank}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 3.$$

$L_4^7 :< x_1, x_1 >= 0, < x_1, x_2 >= b_{12}y, < x_1, x_3 >= b_{13}y, < x_2, x_2 >= 0,$
$< x_2, x_3 >= b_{23}y, < x_3, x_3 >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, b_{12}^2 + b_{13}^2 + b_{23}^2 \neq 0.$
$< x_1, y >= a_{11}x_1 + a_{12}x_2, < x_2, y >= a_{21}x_1 + a_{22}x_2, < x_3, y >= a_{31}x_1 + a_{32}x_2,$

$$\operatorname{rank}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = 2.$$

$L_4^8 :< x_1, x_1 >= 0, < x_1, x_2 >= b_{12}y, < x_1, x_3 >= b_{13}y, < x_2, x_2 >= 0,$
$< x_2, x_3 >= b_{23}y, < x_3, x_3 >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, b_{12}^2 + b_{13}^2 + b_{23}^2 \neq 0.$
$< x_1, y >= a_{11}x_1 + a_{12}x_2 + a_{13}x_3,$
$< x_2, y >= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, < x_3, y >= a_{31}x_1 + a_{32}x_2 + a_{33}x_3,$

$$\operatorname{rank}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 3.$$

$L_4^9 :< x_1, y_1 >= x_1, < x_1, y_2 >= a_{12}x_1, < x_2, y_1 >= kx_1, < x_2, y_2 >=$
$a_{22}x_1, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0.$

$L_4^{10} :< x_1, x_1 >= 0, < x_1, x_2 >= y_1, < x_2, x_2 >= 0, < y_1, y_1 >= 0,$
$< y_1, y_2 >= b_{12}y_1, < y_2, y_2 >= b_{22}y_1, < L_{\bar{0}}, L_{\bar{1}} >= 0.$

$L_4^{11} :< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, < x_1, y_1 >= a_{11}^1 x_1 + a_{11}^2 x_2,$
$< x_1, y_2 >= a_{12}^1 x_1 + a_{12}^2 x_2, < x_2, y_1 >= a_{21}^1 x_1 + a_{21}^2 x_2, < x_2, y_2 >= a_{22}^1 x_1 +$
$a_{22}^2 x_2,$

$$\operatorname{rank}\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{21}^1 & a_{22}^1 \\ a_{11}^2 & a_{12}^2 & a_{21}^2 & a_{22}^2 \end{pmatrix} = 2.$$

$L_4^{12}$ :$< x_1, y_1 >= 0, < x_1, y_2 >= b_{12}x_1, < x_2, y_1 >= 0, < x_2, y_2 >= x_1,$
$< y_1, y_1 >= c_{11}y_1, < y_1, y_2 >= c_{12}y_1, < y_2, y_2 >= y_1, < L_{\bar{0}}, L_{\bar{0}} >= 0.$

$L_4^{13}$ :$< x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y_1, < x_2, x_2 >= 0, < x_1, y_1 >= 0,$
$< x_1, y_2 >= b_{12}x_1, < x_2, y_1 >= b_{21}x_1, < x_2, y_2 >= b_{22}x_1, < y_1, y_1 >= 0,$
$< y_1, y_2 >= b_{12}y_1, < y_2, y_2 >= 0;\ a_{12}^2 + b_{12}^2 \neq 0, b_{12}^2 + b_{21}^2 + b_{22}^2 \neq 0.$

$L_4^{14}$ :$< x_1, y_1 >= 0, < x_1, y_2 >= k_5x_1, < x_2, y_1 >= 0,$
$< x_2, y_2 >= k_6x_1, < y_1, y_1 >= 0, < y_1, y_2 >= y_1, < y_2, y_2 >= 0,$
$< L_{\bar{0}}, L_{\bar{0}} >= 0; k_5^2 + k_6^2 \neq 0.$

$L_4^{15}$ :$< x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y_1, < x_2, x_2 >= 0, < y_1, y_1 >= 0,$
$< y_1, y_2 >= b_{12}y_1, < y_2, y_2 >= 0, a_{12}^2 + b_{12}^2 \neq 0. < x_1, y_1 >= a_{11}^1x_1 + a_{11}^2x_2,$
$< x_1, y_2 >= a_{12}^1x_1 + a_{12}^2x_2, < x_2, y_1 >= a_{21}^1x_1 + a_{21}^2x_2, < x_2, y_2 >= a_{22}^1x_1 +$
$a_{22}^2x_2,$

$$\operatorname{rank}\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{21}^1 & a_{22}^1 \\ a_{11}^2 & a_{12}^2 & a_{21}^2 & a_{22}^2 \end{pmatrix} = 2.$$

$L_4^{16}$ :$< x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y_1, < x_2, x_2 >= 0, < y_1, y_1 >= 0,$
$< y_1, y_2 >= 0, < y_2, y_2 >= b_{22}y_1, < x_1, y_1 >= 0, < x_1, y_2 >= a_{12}^1x_1 +$
$a_{12}^2x_2, < x_2, y_1 >= 0, < x_2, y_2 >= a_{22}^1x_1 + a_{22}^2x_2, a_{12}^2 + b_{22}^2 \neq 0,$

$$\operatorname{rank}\begin{pmatrix} 0 & a_{12}^1 & 0 & a_{22}^1 \\ 0 & a_{12}^2 & 0 & a_{22}^2 \end{pmatrix} = 2.$$

$L_4^{17}$ :$< x, y_1 >= x, < x, y_2 >= 0, < x, y_3 >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0,$
$< L_{\bar{1}}, L_{\bar{1}} >= 0.$

$L_4^{18}$ :$< y_1, y_1 >= 0, < y_1, y_2 >= b_{12}y_1, < y_1, y_3 >= b_{13}y_1, < y_2, y_2 >= 0,$
$< y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0;\ b_{12}^2 + b_{13}^2 +$
$b_{23}^2 \neq 0.$

$L_4^{19}$ :$< y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_1, y_3 >= 0, < y_2, y_2 >= b_{22}y_1,$
$< y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= b_{33}y_1, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0;\ b_{22}^2 +$
$b_{23}^2 + b_{33}^2 \neq 0.$

$L_4^{20} :< y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2, < y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2, < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2, < y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2,$
$< y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0,$

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2 \end{pmatrix} = 2.$$

$L_4^{21} :< x, y_1 >= 0, < x, y_2 >= a_2 x, < x, y_3 >= a_3 x, < y_1, y_1 >= 0,$
$< y_1, y_2 >= 0, < y_1, y_3 >= 0, < y_2, y_2 >= b_{22} y_1, < y_2, y_3 >= b_{23} y_1,$
$< y_3, y_3 >= b_{33} y_1, < L_{\bar{0}}, L_{\bar{0}} >= 0;\ b_{22}^2 + b_{23}^2 + b_{33}^2 \neq 0, a_2^2 + a_3^2 \neq 0.$

$L_4^{22} :< x, y_1 >= 0, < x, y_2 >= a_2 x, < x, y_3 >= a_3 x, < y_1, y_1 >= 0,$
$< y_1, y_2 >= b_{12} y_1, < y_1, y_3 >= b_{13} y_1, < y_2, y_2 >= 0, < y_2, y_3 >= b_{23} y_1,$
$< y_3, y_3 >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0;\ b_{12}^2 + b_{13}^2 + b_{23}^2 \neq 0, a_2^2 + a_3^2 \neq 0.$

$L_4^{23} :< y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2 + a_{11}^3 y_3, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2 + a_{12}^3 y_3,$
$< y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2 + a_{13}^3 y_3, < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2 + a_{22}^3 y_3,$
$< y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2 + a_{23}^3 y_3, < y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2 + a_{33}^3 y_3,$

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2 \\ a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{22}^3 & a_{23}^3 & a_{33}^3 \end{pmatrix} = 3.$$

$L_4^{24} :< x, y_1 >= a_1 x, < x, y_2 >= a_2 x, < x, y_3 >= a_3 x, < y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2,$
$< y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2, < y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2, < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2, < y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2, < y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2, < L_{\bar{0}}, L_{\bar{0}} >= 0, a_1^2 + a_2^2 + a_3^2 \neq 0,$

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2 \end{pmatrix} = 2.$$

$L_4^{25} :< x, y_1 >= a_1 x, < x, y_2 >= a_2 x, < x, y_3 >= a_3 x, < y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2 + a_{11}^3 y_3, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2 + a_{12}^3 y_3, < y_1, y_3 >= a_{13}^1 y_1 +$

$a^2_{13}y_2 + a^3_{13}y_3, < y_2, y_2 >= a^1_{22}y_1 + a^2_{22}y_2 + a^3_{22}y_3, < y_2, y_3 >= a^1_{23}y_1 + a^2_{23}y_2 +$
$a^3_{23}y_3, < y_3, y_3 >= a^1_{33}y_1 + a^2_{33}y_2 + a^3_{33}y_3, < L_{\bar{0}}, L_{\bar{0}} >= 0;\ a^2_1 + a^2_2 + a^2_3 \neq 0,$

$$\operatorname{rank}\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{22} & a^1_{23} & a^1_{33} \\ a^2_{11} & a^2_{12} & a^2_{13} & a^2_{22} & a^2_{23} & a^2_{33} \\ a^3_{11} & a^3_{12} & a^3_{13} & a^3_{22} & a^3_{23} & a^3_{33} \end{pmatrix} = 3.$$

$L^{26}_4 :< y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_1, y_3 >= 0, < y_1, y_4 >= 0,$
$< y_2, y_2 >= b_{22}y_1, < y_2, y_3 >= b_{23}y_1, < y_2, y_4 >= b_{24}y_1, < y_3, y_3 >= b_{33}y_1,$
$< y_3, y_4 >= b_{34}y_1, < y_4, y_4 >= b_{44}y_1;\ b^2_{22} + b^2_{23} + b^2_{24} + b^2_{33} + b^2_{34} + b^2_{44} \neq 0.$

$L^{27}_4 :< y_1, y_1 >= 0, < y_1, y_2 >= b_{12}y_1, < y_1, y_3 >= b_{13}y_1, < y_1, y_4 >=$
$b_{14}y_1, < y_2, y_2 >= 0, < y_2, y_3 >= b_{23}y_1, < y_2, y_4 >= b_{24}y_1, < y_3, y_3 >= 0,$
$< y_3, y_4 >= b_{34}y_1, < y_4, y_4 >= 0;\ b^2_{12} + b^2_{13} + b^2_{14} + b^2_{23} + b^2_{24} + b^2_{34} \neq 0.$

$L^{28}_4 :< y_1, y_1 >= a^1_{11}y_1 + a^2_{11}y_2, < y_1, y_2 >= a^1_{12}y_1 + a^2_{12}y_2, < y_1, y_3 >=$
$a^1_{13}y_1 + a^2_{13}y_2, < y_1, y_4 >= a^1_{14}y_1 + a^2_{14}y_2, < y_2, y_2 >= a^1_{22}y_1 + a^2_{22}y_2, < y_2, y_3 >=$
$a^1_{23}y_1 + a^2_{23}y_2, < y_2, y_4 >= a^1_{24}y_1 + a^2_{24}y_2, < y_3, y_3 >= a^1_{33}y_1 + a^2_{33}y_2,$
$< y_3, y_4 >= a^1_{34}y_1 + a^2_{34}y_2, < y_4, y_4 >= a^1_{44}y_1 + a^2_{44}y_2,$

$$\operatorname{rank}\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{14} & a^1_{22} & a^1_{23} & a^1_{24} & a^1_{33} & a^1_{34} & a^1_{44} \\ a^2_{11} & a^2_{12} & a^2_{13} & a^2_{14} & a^2_{22} & a^2_{23} & a^2_{24} & a^2_{33} & a^2_{34} & a^2_{44} \end{pmatrix} = 2.$$

$L^{29}_4 :< y_1, y_1 >= a^1_{11}y_1 + a^2_{11}y_2 + a^3_{11}y_3, < y_1, y_2 >= a^1_{12}y_1 + a^2_{12}y_2 + a^3_{12}y_3,$
$< y_1, y_3 >= a^1_{13}y_1 + a^2_{13}y_2 + a^3_{13}y_3, < y_1, y_4 >= a^1_{14}y_1 + a^2_{14}y_2 + a^3_{14}y_3,$
$< y_2, y_2 >= a^1_{22}y_1 + a^2_{22}y_2 + a^3_{22}y_3, < y_2, y_3 >= a^1_{23}y_1 + a^2_{23}y_2 + a^3_{23}y_3,$
$< y_2, y_4 >= a^1_{24}y_1 + a^2_{24}y_2 + a^3_{24}y_3, < y_3, y_3 >= a^1_{33}y_1 + a^2_{33}y_2 + a^3_{33}y_3,$
$< y_3, y_4 >= a^1_{34}y_1 + a^2_{34}y_2 + a^3_{34}y_3, < y_4, y_4 >= a^1_{44}y_1 + a^2_{44}y_2 + a^3_{44}y_3,$

$$\operatorname{rank}\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{14} & a^1_{22} & a^1_{23} & a^1_{24} & a^1_{33} & a^1_{34} & a^1_{44} \\ a^2_{11} & a^2_{12} & a^2_{13} & a^2_{14} & a^2_{22} & a^2_{23} & a^2_{24} & a^2_{33} & a^2_{34} & a^2_{44} \\ a^3_{11} & a^3_{12} & a^3_{13} & a^3_{14} & a^3_{22} & a^3_{23} & a^3_{24} & a^3_{33} & a^3_{34} & a^3_{44} \end{pmatrix} = 3.$$

$L_4^{30} :< y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2 + a_{11}^3 y_3 + a_{11}^4 y_4, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2 + a_{12}^3 y_3 + a_{12}^4 y_4,$
$< y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2 + a_{13}^3 y_3 + a_{13}^4 y_4, < y_1, y_4 >= a_{14}^1 y_1 + a_{14}^2 y_2 + a_{14}^3 y_3 + a_{14}^4 y_4,$
$< y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2 + a_{22}^3 y_3 + a_{22}^4 y_4, < y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2 + a_{23}^3 y_3 + a_{23}^4 y_4,$
$< y_2, y_4 >= a_{24}^1 y_1 + a_{24}^2 y_2 + a_{24}^3 y_3 + a_{24}^4 y_4, < y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2 + a_{33}^3 y_3 + a_{33}^4 y_4,$
$< y_3, y_4 >= a_{34}^1 y_1 + a_{34}^2 y_2 + a_{34}^3 y_3 + a_{34}^4 y_4, < y_4, y_4 >= a_{44}^1 y_1 + a_{44}^2 y_2 + a_{44}^3 y_3 + a_{44}^4 y_4,$

$$\operatorname{rank}\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 & a_{22}^1 & a_{23}^1 & a_{24}^1 & a_{33}^1 & a_{34}^1 & a_{44}^1. \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{14}^2 & a_{22}^2 & a_{23}^2 & a_{24}^2 & a_{33}^2 & a_{34}^2 & a_{44}^2 \\ a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{14}^3 & a_{22}^3 & a_{23}^3 & a_{24}^3 & a_{33}^3 & a_{34}^3 & a_{44}^3 \\ a_{11}^4 & a_{12}^4 & a_{13}^4 & a_{14}^4 & a_{22}^4 & a_{23}^4 & a_{24}^4 & a_{33}^4 & a_{34}^4 & a_{44}^4 \end{pmatrix} = 4.$$

*Proof.* *1.* Let $L$ be a 1-dimensional 2-Lie superalgebra.

*(1)* Assume that $\dim L_{\bar{0}} = 1$. Then $< L, L >= 0$.

*(2)* Assume that $\dim L_{\bar{1}} = 1$. Then there is a basis $\{y\}$ in $L_{\bar{1}}$. By Jacobi identities, we have $< y, y >= ay$, then $a = 0$, hence $< L, L >= 0$. Thus we obtain type

$$L_1^1 :< L, L >= 0.$$

*2.* Let $L$ be a 2-dimensional 2-Lie superalgebra.

*(1)* Assume that $\dim L_{\bar{0}} = 2$. Then $< L, L >= 0$. Thus we obtain type

$$L_2^1 :< L, L >= 0.$$

*(2)* Assume that $\dim L_{\bar{0}} = 1$ and $\dim L_{\bar{1}} = 1$. Then there is a basis $\{x, y\}$ in $L$, where $x \in L_{\bar{0}}$ and $y \in L_{\bar{1}}$. We classify $L$ by the dimension of $< L, L >$ as the following cases.

*A.* $\dim < L, L >= 0$. Then $< L, L >= 0$, which is the type $L_2^1$.

*B.* $\dim < L, L >= 1$.

a. If $< L, L >= \mathrm{F}x$, then we have $< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, < x, y >= ax (a \neq 0)$. Replace $y$ by $a^{-1}y$, then we have

$$L_2^2 :< x, y >= x, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0.$$

b. If $< L, L >= \mathrm{F}y$, we have $< L_{\bar{0}}, L_{\bar{1}} >= 0, < x, x >= by, < y, y >= ay$. By the definition of 2-Lie superalgebras, we have $b = 0$. By Jacobi identities, we have $a = 0$. Hence $< L_{\bar{0}}, L_{\bar{0}} >= 0$ and $< L_{\bar{1}}, L_{\bar{1}} >= 0$, thus $< L, L >= 0$, which is the type $L_2^1$.

*C.* $\dim < L, L >= 2$. Then $< L, L >= \mathrm{F}x + \mathrm{F}y$, and so we have $< x, x >= ay, < x, y >= bx, < y, y >= cy$. By the definition of 2-Lie superalgebras, we have $a = 0$. By Jacobi identities, we have $c = 0$, which contradicts $\dim < L, L >= 2$. So we claim that there is no 2-Lie superalgebra in this case.

*(3)* Assume that $\dim L_{\bar{1}} = 2$. Then there is a basis $\{y_1, y_2\}$ in $L_{\bar{1}}$. We classify $L$ by the dimension of $< L, L >$ as the following cases.

*A.* $\dim < L, L >= 0$. Then $< L, L >= 0$.

*B.* $\dim < L, L >= 1$. Then $< L, L >= \mathrm{F}y_1$, and so we have $< y_1, y_1 >= a_1 y_1, < y_1, y_2 >= a_2 y_1, < y_2, y_2 >= a_3 y_1$. By Jacobi identities, we have $a_1 = 0, a_2 a_3 = 0$. Note that $< L, L >= \mathrm{F}y_1$, then there is at least one nonzero element in the set $\{a_2, a_3\}$.

*Case 1* If $a_2 = 0, a_3 \neq 0$, then replace $y_1$ by $a_3 y_1$, thus we have

$$L_2^3 :< y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_2, y_2 >= y_1.$$

*Case 2* If $a_2 \neq 0, a_3 = 0$, then replace $y_2$ by $a_2^{-1} y_2$, thus we have

$$L_2^4 :< y_1, y_1 >= 0, < y_1, y_2 >= y_1, < y_2, y_2 >= 0.$$

*C.* $\dim < L, L >= 2$. Then $< L, L >= \mathrm{F}y_1 + \mathrm{F}y_2$, and we have
$< L_{\bar{0}}, L_{\bar{1}} >= 0, < y_1, y_1 >= a_{11}y_1 + a_{12}y_2, < y_1, y_2 >= a_{21}y_1 + a_{22}y_2,$
$< y_2, y_2 >= a_{31}y_1 + a_{32}y_2$, $\mathrm{rank}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = r.$

If $r = 1$, then $\dim < L, L >= 1$, which contradicts $\dim < L, L >= 2$. Hence $r = 2$. By Jacobi identities, we have

$$a_{11}^2 + a_{12}a_{21} = 0, \quad a_{11}a_{21} + a_{12}a_{31} = 0,$$

$$a_{12}a_{11} + a_{12}a_{22} = 0, \quad a_{11}a_{22} + a_{12}a_{32} = 0,$$

$$a_{32}^2 + a_{22}a_{31} = 0, \quad a_{11}a_{31} + a_{21}a_{32} = 0,$$

$$a_{31}a_{21} + a_{31}a_{32} = 0, \quad a_{12}a_{31} + a_{22}a_{32} = 0.$$

If $a_{11} = 0$, then $a_{12}a_{21} = 0, a_{12}a_{22} = 0, a_{12}a_{31} = 0, a_{12}a_{32} = 0, a_{21}a_{32} = 0$. So we have $a_{12} = a_{32} = 0$ or $a_{21} = a_{22} = a_{31} = a_{32} = 0$. Since $r = 2$, we have $a_{12} = a_{32} = 0$. Hence $a_{22} \neq 0$. If $a_{22}a_{31} = 0$, then $a_{31} = 0$, which contradicts $r = 2$. So $a_{11} \neq 0$. If $a_{12} = 0$, then $a_{12} = 0$, which is a contradiction. Then $a_{12} \neq 0$. Similarly, $a_{22} \neq 0, a_{31} \neq 0, a_{32} \neq 0$. Hence we have $\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} = \frac{a_{31}}{a_{32}}$, which contradicts $r = 2$. Hence we claim that there is no 2-Lie superalgebra in this case.

*3.* Let $L$ be a 3-dimensional 2-Lie superalgebra.

*(1)* Assume that $\dim L_{\bar{0}} = 3$. Then $< L, L >= 0$. Thus we obtain type

$$L_3^1 : < L, L >= 0.$$

*(2)* Assume that $\dim L_{\bar{0}} = 1$ and $\dim L_{\bar{1}} = 2$. Then there is a basis $\{x, y_1, y_2\}$ of $L$, where $x \in L_{\bar{0}}$ and $y_1, y_2 \in L_{\bar{1}}$. We classify $L$ by the dimension of $< L, L >$ as the following cases.

*A.* $\dim < L, L >= 0$. Then $< L, L >= 0$.

*B.* $\dim < L, L >= 1$.

a. If $< L, L >= Fx$, then we have $< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0$, $< x, y_1 >= a_1 x, < x, y_2 >= a_2 x$. Without loss of generality we may suppose that $a_1 \neq 0$. Replace $y_1$, $y_2$ by $a_1^{-1} y_1$, $y_2 - \frac{a_2}{a_1} y_1$ respectively, then one gets $< x, y_1 >= x, < x, y_2 >= 0$. Thus we have

$$L_3^2 : < x, y_1 >= x, < x, y_2 >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0.$$

b. If $< L, L >= Fy_1$, then we have $< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0$, $< y_1, y_1 >= a_1 y_1, < y_1, y_2 >= a_2 y_1, < y_2, y_2 >= a_3 y_1$. By Jacobi identities, we have $a_1 = 0, a_2 a_3 = 0$. Note that $< L, L >= Fy_1$, then there is at least one nonzero element in $\{a_2, a_3\}$.

*Case 1.* If $a_2 = 0, a_3 \neq 0$, then replace $y_1$ by $a_3 y_1$, thus we have

$$\begin{aligned} L_3^3 \ : \ & < y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_2, y_2 >= y_1, \\ & < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0. \end{aligned}$$

*Case 2.* If $a_2 \neq 0, a_3 = 0$, then replace $y_2$ by $a_2^{-1} y_2$, thus we have

$$\begin{aligned} L_3^4 \ : \ & < y_1, y_1 >= 0, < y_1, y_2 >= y_1, < y_2, y_2 >= 0, \\ & < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0. \end{aligned}$$

*C.* $\dim < L, L >= 2$.

a. If $< L, L >= \mathrm{F}y_1 + \mathrm{F}y_2$, then we have $< L_{\bar{0}}, L_{\bar{0}} >= 0, < y_1, y_1 >= a_{11} y_1 + a_{12} y_2, < y_1, y_2 >= a_{21} y_1 + a_{22} y_2$, $< y_2, y_2 >= a_{31} y_1 + a_{32} y_2$. $\mathrm{rank} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = r$.

If $r = 1$, then $\dim < L, L >= 1$, which contradicts $\dim < L, L >= 2$. Hence $r = 2$. By Jacobi identities, we have

$$a_{11}^2 + a_{12} a_{21} = 0, \quad a_{11} a_{21} + a_{12} a_{31} = 0,$$

$$a_{12}a_{11} + a_{12}a_{22} = 0, \quad a_{11}a_{22} + a_{12}a_{32} = 0,$$
$$a_{32}^2 + a_{22}a_{31} = 0, \quad a_{11}a_{31} + a_{21}a_{32} = 0,$$
$$a_{31}a_{21} + a_{31}a_{32} = 0, \quad a_{12}a_{31} + a_{22}a_{32} = 0.$$

If $a_{11} = 0$, then $a_{12}a_{21} = 0, a_{12}a_{22} = 0, a_{12}a_{31} = 0, a_{12}a_{32} = 0, a_{21}a_{32} = 0$. Then $a_{12} = a_{32} = 0$ or $a_{21} = a_{22} = a_{31} = a_{32} = 0$. Since $r = 2$, we have $a_{12} = a_{32} = 0$. Hence $a_{22} \neq 0$. If $a_{22}a_{31} = 0$, then $a_{31} = 0$, which contradicts $r = 2$. So $a_{11} \neq 0$. If $a_{12} = 0$, then $a_{12} = 0$, a contradiction. Then $a_{12} \neq 0$. Similarly $a_{22} \neq 0$,$a_{31} \neq 0$,$a_{32} \neq 0$. Hence we have $\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} = \frac{a_{31}}{a_{32}}$, which contradicts $r = 2$. So we claim that there is no 2-Lie superalgebra in this case.

b. If $< L, L >= \mathrm{F}x + \mathrm{F}y_1$, then we have $< L_{\bar{0}}, L_{\bar{0}} >= 0$, $< x, y_1 >= a_1x$, $< x, y_2 >= a_2x$, $< y_1, y_1 >= b_1y_1$, $< y_1, y_2 >= b_2y_1$, $< y_2, y_2 >= b_3y_1$. By Jacobi identities, we have $b_1 = 0, a_1b_2 = 0, a_1b_3 = 0, b_2b_3 = 0$. Then $a_1 = b_2 = 0$ or $a_1 = b_3 = 0$ or $b_2 = b_3 = 0$.

*Case 1.* If $a_1 = b_2 = 0$, then $< x, y_1 >= 0, < x, y_2 >= a_2x, < y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_2, y_2 >= b_3y_1$. Since $< L, L >= Fx + Fy_1$, we have $a_2 \neq 0, b_3 \neq 0$. Replace $y_2$, $k$ by $a_2^{-1}y_2$, $\frac{b_3}{a_2^2}$ respectively, then we have

$$\begin{aligned} L_3^5 \ : \ & < y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_2, y_2 >= ky_1, \\ & < x, y_1 >= 0, < x, y_2 >= x, < L_{\bar{0}}, L_{\bar{0}} >= 0. \end{aligned}$$

*Case 2.* If $a_1 = b_3 = 0$, then $< x, y_1 >= 0, < x, y_2 >= a_2x, < y_1, y_1 >= 0, < y_1, y_2 >= b_2y_1, < y_2, y_2 >= 0$. Replace $y_2$, $k_1$ by $a_2^{-1}y_2$, $\frac{b_2}{a_2}$ respectively, then we have

$$\begin{aligned} L_3^6 \ : \ & < y_1, y_1 >= 0, < y_1, y_2 >= k_1y_1, < y_2, y_2 >= 0, \\ & < x, y_1 >= 0, < x, y_2 >= x, < L_{\bar{0}}, L_{\bar{0}} >= 0. \end{aligned}$$

*Case 3.* If $b_2 = b_3 = 0$, then $< L_{\bar{1}}, L_{\bar{1}} >= 0$, which contradicts $< L, L >= Fx + Fy_1$. So we claim that there is no 2-Lie superalgebra in this case.

*D.* $\dim < L, L >= 3$, $< L, L >= \mathrm{F}x+\mathrm{F}y_1+\mathrm{F}y_2$. Then we have $< x, y_1 >= a_1x$, $< x, y_2 >= a_2x$, $< y_1, y_1 >= a_{11}y_1 + a_{12}y_2$, $< y_1, y_2 >= a_{21}y_1 + a_{22}y_2$, $< y_2, y_2 >= a_{31}y_1 + a_{32}y_2$, $\mathrm{rank}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = r$.

If $r = 1$, then $\dim < L, L >= 1$, which contradicts $\dim < L, L >= 2$. Hence $r = 2$. By Jacobi identities, we have

$$a_{11}^2 + a_{12}a_{21} = 0, \quad a_{11}a_{21} + a_{12}a_{31} = 0,$$

$$a_{12}a_{11} + a_{12}a_{22} = 0, \quad a_{11}a_{22} + a_{12}a_{32} = 0,$$

$$a_{32}^2 + a_{22}a_{31} = 0, \quad a_{11}a_{31} + a_{21}a_{32} = 0,$$

$$a_{31}a_{21} + a_{31}a_{32} = 0, \quad a_{12}a_{31} + a_{22}a_{32} = 0.$$

If $a_{11} = 0$, then $a_{12}a_{21} = 0, a_{12}a_{22} = 0, a_{12}a_{31} = 0, a_{12}a_{32} = 0, a_{21}a_{32} = 0$. Thus either $a_{12} = a_{32} = 0$ or $a_{21} = a_{22} = a_{31} = a_{32} = 0$. Since $r = 2$, we have $a_{12} = a_{32} = 0$. Hence $a_{22} \neq 0$. If $a_{22}a_{31} = 0$, then $a_{31} = 0$, which contradicts $r = 2$. So $a_{11} \neq 0$. If $a_{12} = 0$, then $a_{12} = 0$, which is a contradiction. Then $a_{12} \neq 0$. Similarly $a_{22} \neq 0$,$a_{31} \neq 0$,$a_{32} \neq 0$. Hence we have $\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} = \frac{a_{31}}{a_{32}}$, which contradicts $r = 2$. So we claim that there is no 2-Lie superalgebra in this case.

*(3)* Assume that $\dim L_{\bar{0}} = 2$ and $\dim L_{\bar{1}} = 1$. Then there is a basis $\{x_1, x_2, y\}$ of $L$, where $x_1, x_2 \in L_{\bar{0}}$ and $y \in L_{\bar{1}}$. We classify $L$ by the dimension of $< L, L >$ as the following cases.

*A.* $\dim < L, L >= 0$. Then $< L, L >= 0$.

*B.* $\dim < L, L >= 1$.

a. If $< L, L >= \mathrm{F}x_1$, then we have $< L_{\bar{0}}, L_{\bar{0}} >= 0$, $< L_{\bar{1}}, L_{\bar{1}} >= 0$, $< x_1, y >= a_1x_1, < x_2, y >= a_2x_1$. Without loss of generality we may suppose that $a_1 \neq 0$, then the replacements of $y$ by $a_1^{-1}y$ and of $x_2$ by

$\dot{x}_2 - \frac{a_2}{a_1}x_1$ give $< x_1, y >= x_1$ and $< x_2, y >= 0$. Thus we have

$$L_3^7 :< x_1, y >= x_1, < x_2, y >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0.$$

b. If $< L, L >= \mathrm{F}y$, then we have $< L_{\bar{0}}, L_{\bar{1}} >= 0$, $< x_1, x_1 >= a_{11}y, < x_1, x_2 >= a_{12}y, < x_2, x_2 >= a_{22}y, < y, y >= by$. By Jacobi identities, we have $b = 0$. It's clear $< L_{\bar{1}}, L_{\bar{1}} >= 0$, $a_{11} = a_{22} = 0$. Hence $a_{12} \neq 0$. Then the replacement of $y$ by $a_{12}y$ gives $< x_1, x_2 >= y$. Thus we have

$$\begin{aligned} L_3^8 \ : \ & < x_1, x_1 >= 0, < x_1, x_2 >= y, < x_2, x_2 >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, \\ & < L_{\bar{0}}, L_{\bar{1}} >= 0. \end{aligned}$$

*C.* $\dim < L, L >= 2$.

a. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}y$, then we have $< x_1, x_1 >= a_{11}y, < x_1, x_2 >= a_{12}y, < x_2, x_2 >= a_{22}y$, $< x_1, y >= b_1x_1, < x_2, y >= b_2x_1, < y, y >= cy$. By Jacobi identities, we have $a_{11} = 0, a_{22} = 0, c = 0, a_{12}b_1 = 0$. Note that $< L, L >= \mathrm{F}x_1 + \mathrm{F}y$, then $a_{11} = a_{22} = b_1 = 0$. Then the replacements of $y$ by $a_{12}y$ and of $k_4$ by $a_{12}b_2$ give $< x_1, x_2 >= y$ and $< x_2, y >= k_4x_1$. Thus we have

$$\begin{aligned} L_3^9 \ : \ & < x_1, x_1 >= 0, < x_1, x_2 >= y, < x_2, x_2 >= 0, < x_1, y >= 0, \\ & < x_2, y >= k_4x_1, < L_{\bar{1}}, L_{\bar{1}} >= 0. \end{aligned}$$

b. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2$, then $< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, < x_1, y >= b_{11}x_1 + b_{12}x_2, < x_2, y >= b_{21}x_1 + b_{22}x_2$. Thus we have

$$\begin{aligned} L_3^{10} \ : \ & < x_1, y >= b_{11}x_1 + b_{12}x_2, < x_2, y >= b_{21}x_1 + b_{22}x_2, < L_{\bar{0}}, L_{\bar{0}} >= 0, \\ & < L_{\bar{1}}, L_{\bar{1}} >= 0, \mathrm{rank} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = 2. \end{aligned}$$

*D.* $\dim < L, L >= 3$, $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2 + \mathrm{F}y$. Then we have $< x_1, x_1 >= a_{11}y, < x_1, x_2 >= a_{12}y, < x_2, x_2 >= a_{22}y, < x_1, y >= b_{11}x_1 +$

$b_{12}x_2, < x_2, y >= b_{21}x_1 + b_{22}x_2, < y, y >= cy$. By Jacobi identities, we have

$$a_{11}b_{11} = 0, a_{11}b_{21} = 0, a_{11}b_{22} = 0, a_{22}b_{11} = 0,$$

$$a_{22}b_{21} = 0, a_{22}b_{22} = 0, a_{12}(b_{11} + b_{22}) = 0, c = 0.$$

Thus either $a_{11} = a_{22} = 0$ or $b_{11} = b_{21} = b_{22} = 0$. Since $\dim < L, L >= 3$, we have $a_{12} \neq 0$, $b_{22} = -b_{11}$, thus we have

$$\begin{aligned} L_3^{11} \ : \ & < x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y, < x_2, x_2 >= 0, \\ & < x_1, y >= b_{11}x_1 + b_{12}x_2, < x_2, y >= b_{21}x_1 - b_{11}x_2, < L_{\bar{1}}, L_{\bar{1}} >= 0; \\ & a_{12} \neq 0, \mathrm{rank} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{pmatrix} = 2. \end{aligned}$$

*(4)* Assume that $\dim L_{\bar{1}} = 3$. Then there is a basis $\{y_1, y_2, y_3\} \subseteq L_{\bar{1}}$. We classify $L$ by the dimension of $< L, L >$ as the following cases.

*A.* $\dim < L, L >= 0$. Then $< L, L >= 0$.

*B.* $\dim < L, L >= 1$, $< L, L >= \mathrm{F}y_1$. Then we have $< y_1, y_1 >= b_{11}y_1$, $< y_1, y_2 >= b_{12}y_1, < y_1, y_3 >= b_{13}y_1, < y_2, y_2 >= b_{22}y_1, < y_2, y_3 >= b_{23}y_1$, $< y_3, y_3 >= b_{33}y_1$. By Jacobi identities, we have $b_{11} = 0, b_{12}b_{22} = 0, b_{12}b_{33} = 0, b_{13}b_{22} = 0, b_{13}b_{33} = 0$. Thus either $b_{22} = b_{33} = 0$ or $b_{12} = b_{13} = 0$.

*Case 1.* If $b_{22} = b_{33} = 0$, then we have

$$\begin{aligned} L_3^{12} \ : \ & < y_1, y_1 >= 0, < y_1, y_2 >= b_{12}y_1, < y_1, y_3 >= b_{13}y_1, < y_2, y_2 >= 0, \\ & < y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= 0; b_{12}^2 + b_{13}^2 + b_{23}^2 \neq 0. \end{aligned}$$

*Case 2.* If $b_{12} = b_{13} = 0$, then we have

$$\begin{aligned} L_3^{13} \ : \ & < y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_1, y_3 >= 0, < y_2, y_2 >= b_{22}y_1, \\ & < y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= b_{33}y_1; b_{22}^2 + b_{23}^2 + b_{33}^2 \neq 0. \end{aligned}$$

*C.* $\dim < L, L >= 2$, $< L, L >= Fy_1 + Fy_2$. Then we have $< y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2$, $< y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2$, $< y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2$, $<$

$y_2, y_2 >= a^1_{22}y_1 + a^2_{22}y_2, < y_2, y_3 >= a^1_{23}y_1 + a^2_{23}y_2, < y_3, y_3 >= a^1_{33}y_1 + a^2_{33}y_2,$ and

$$\operatorname{rank}\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{22} & a^1_{23} & a^1_{33} \\ a^2_{11} & a^2_{12} & a^2_{13} & a^2_{22} & a^2_{23} & a^2_{33} \end{pmatrix} = r.$$

*D.* $\dim < L, L >= 3$, $< L, L >= \mathrm{F}y_1 + \mathrm{F}y_2 + \mathrm{F}y_3$. Then we have $< y_1, y_1 >= a^1_{11}y_1 + a^2_{11}y_2 + a^3_{11}y_3, < y_1, y_2 >= a^1_{12}y_1 + a^2_{12}y_2 + a^3_{12}y_3, < y_1, y_3 >= a^1_{13}y_1 + a^2_{13}y_2 + a^3_{13}y_3, < y_2, y_2 >= a^1_{22}y_1 + a^2_{22}y_2 + a^3_{22}y_3, < y_2, y_3 >= a^1_{23}y_1 + a^2_{23}y_2 + a^3_{23}y_3, < y_3, y_3 >= a^1_{33}y_1 + a^2_{33}y_2 + a^3_{33}y_3,$ and

$$\operatorname{rank}\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{13} & a^1_{22} & a^1_{23} & a^1_{33} \\ a^2_{11} & a^2_{12} & a^2_{13} & a^2_{22} & a^2_{23} & a^2_{33} \\ a^3_{11} & a^3_{12} & a^3_{13} & a^3_{22} & a^3_{23} & a^3_{33} \end{pmatrix} = r.$$

If $r = 1$, then $\dim < L, L >= 1$, which contradicts $\dim < L, L >= 2$. If $r = 2$, then $\dim < L, L >= 2$, and this contradicts $\dim < L, L >= 3$. Then $r = 3$. By Jacobi identities, we have

$$a^1_{11}a^1_{11} + a^2_{11}a^1_{12} + a^3_{11}a^1_{13} = 0, \quad a^1_{12}a^1_{22} + a^1_{22}a^2_{22} + a^3_{22}a^1_{23} = 0.$$

$$a^1_{11}a^2_{11} + a^2_{11}a^2_{12} + a^3_{11}a^2_{13} = 0, \quad a^1_{22}a^2_{12} + a^2_{22}a^2_{22} + a^3_{22}a^2_{23} = 0.$$

$$a^1_{11}a^3_{11} + a^2_{11}a^3_{12} + a^3_{11}a^3_{13} = 0, \quad a^3_{12}a^1_{22} + a^2_{22}a^3_{22} + a^3_{22}a^3_{23} = 0.$$

$$a^1_{11}a^1_{12} + a^2_{11}a^1_{22} + a^3_{11}a^1_{23} = 0, \quad a^1_{13}a^1_{33} + a^1_{23}a^2_{33} + a^1_{33}a^3_{33} = 0.$$

$$a^1_{11}a^2_{12} + a^2_{11}a^2_{22} + a^3_{11}a^2_{23} = 0, \quad a^2_{13}a^1_{33} + a^2_{23}a^2_{33} + a^2_{33}a^3_{33} = 0.$$

$$a^1_{11}a^3_{12} + a^2_{11}a^3_{22} + a^3_{11}a^3_{23} = 0, \quad a^3_{13}a^1_{33} + a^3_{23}a^2_{33} + a^3_{33}a^3_{33} = 0.$$

$$a^1_{11}a^1_{13} + a^2_{11}a^1_{23} + a^3_{11}a^1_{33} = 0, \quad a^1_{22}a^1_{11} + a^2_{22}a^1_{12} + a^3_{22}a^1_{13} = 0.$$

$$a^1_{11}a^2_{13} + a^2_{11}a^2_{23} + a^3_{11}a^2_{33} = 0, \quad a^1_{22}a^2_{11} + a^2_{22}a^2_{12} + a^3_{22}a^2_{13} = 0.$$

$$a^1_{11}a^3_{13} + a^2_{11}a^3_{23} + a^3_{11}a^3_{33} = 0, \quad a^1_{22}a^3_{11} + a^2_{22}a^3_{12} + a^3_{22}a^3_{13} = 0.$$

$$a^1_{22}a^1_{13}+a^2_{22}a^1_{23}+a^3_{22}a^1_{33}=0,\quad a^1_{33}a^1_{11}+a^2_{33}a^1_{12}+a^3_{33}a^1_{13}=0.$$
$$a^1_{22}a^2_{13}+a^2_{22}a^2_{23}+a^3_{22}a^2_{33}=0,\quad a^1_{33}a^2_{11}+a^2_{33}a^2_{12}+a^3_{33}a^2_{13}=0.$$
$$a^1_{22}a^3_{13}+a^2_{22}a^3_{23}+a^3_{22}a^3_{33}=0,\quad a^1_{33}a^3_{11}+a^2_{33}a^3_{12}+a^3_{33}a^3_{13}=0.$$
$$a^1_{33}a^1_{13}+a^2_{33}a^1_{23}+a^3_{33}a^1_{33}=0,\quad a^1_{33}a^2_{13}+a^2_{33}a^2_{23}+a^3_{33}a^2_{33}=0.$$
$$a^1_{33}a^3_{13}+a^2_{33}a^3_{23}+a^3_{33}a^3_{33}=0,$$
$$(a^1_{11}+a^1_{12}-a^3_{13})a^1_{23}+(a^1_{13}-a^2_{23})a^1_{12}-a^2_{13}a^1_{22}+a^3_{12}a^1_{33}=0.$$

Thus we obtain

$$\begin{aligned}L_3^{15}\ :\ &<y_1,y_1>=a^1_{11}y_1+a^2_{11}y_2+a^3_{11}y_3,<y_1,y_2>=a^1_{12}y_1+a^2_{12}y_2+a^3_{12}y_3,\\ &<y_1,y_3>=a^1_{13}y_1+a^2_{13}y_2+a^3_{13}y_3,<y_2,y_2>=a^1_{22}y_1+a^2_{22}y_2+a^3_{22}y_3,\\ &<y_2,y_3>=a^1_{23}y_1+a^2_{23}y_2+a^3_{23}y_3,<y_3,y_3>=a^1_{33}y_1+a^2_{33}y_2+a^3_{33}y_3,\\ &\operatorname{rank}\begin{pmatrix}a^1_{11}&a^1_{12}&a^1_{13}&a^1_{22}&a^1_{23}&a^1_{33}\\ a^2_{11}&a^2_{12}&a^2_{13}&a^2_{22}&a^2_{23}&a^2_{33}\\ a^3_{11}&a^3_{12}&a^3_{13}&a^3_{22}&a^3_{23}&a^3_{33}\end{pmatrix}=3.\end{aligned}$$

*4.* Let $L$ be a 4-dimensional 2-Lie superalgebra.

*(1)* Assume that $\dim L_{\bar{0}}=4$. Then $<L,L>=0$. Thus we have

$$L_4^1:<L,L>=0.$$

*(2)* Assume that $\dim L_{\bar{0}}=3, \dim L_{\bar{1}}=1$. Then there is a basis $\{x_1,x_2,x_3,y\}$ in $L$, where $x_1,x_2,x_3\in L_{\bar{0}}$ and $y\in L_{\bar{1}}$. We classify $L$ by the dimension of $<L,L>$ as the following cases.

*A.* $\dim <L,L>=0$. Then $<L,L>=0$.

*B.* $\dim <L,L>=1$, $<L,L>=\mathrm{F}x_1$.

a. If $<L,L>=\mathrm{F}x_1$, then we have $<L_{\bar{0}},L_{\bar{0}}>=0$, $<L_{\bar{1}},L_{\bar{1}}>=0$, $<x_1,y>=a_1x_1$, $<x_2,y>=a_2x_1$, $<x_3,y>=a_3x_1$. Without loss of generality we may suppose that $a_2\neq 0$, then the replacements of $x_2$ by

$a_2^{-1}x_2$ and of $x_3$ by $x_3 - \frac{a_3}{a_2}x_2$ give $< x_2, y >= x_1$ and $< x_3, y >= 0$, thus we obtain

$$\begin{aligned} L_4^2 \ : \ & < x_1, y >= a_1 x_1, < x_2, y >= x_1, < x_3, y >= 0, \\ & < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0. \end{aligned}$$

b. If $< L, L >= Fy$, then we have $< L_{\bar{1}}, L_{\bar{1}} >= 0$, $< L_{\bar{0}}, L_{\bar{1}} >= 0$, $< x_1, x_1 >= a_{11}y, < x_1, x_2 >= a_{12}y$, $< x_1, x_3 >= a_{13}y, < x_2, x_2 >= a_{22}y, < x_2, x_3 >= a_{23}y, < x_3, x_3 >= a_{33}y$. By the definition of 2-Lie superalgebras, we have $a_{11} = a_{22} = a_{33} = 0$. Thus we obtain

$$\begin{aligned} L_4^3 \ : \ & < L_{\bar{1}}, L_{\bar{1}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0, \\ & < x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y, < x_1, x_3 >= a_{13}y, \\ & < x_2, x_2 >= 0, < x_2, x_3 >= a_{23}y, < x_3, x_3 >= 0; a_{12}^2 + a_{13}^2 + a_{23}^2 \neq 0. \end{aligned}$$

*C.* $\dim < L, L >= 2$.

a. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2$, then we have

$$\begin{aligned} L_4^4 \ : \ & < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, \\ & < x_1, y >= a_{11}x_1 + a_{12}x_2, < x_2, y >= a_{21}x_1 + a_{22}x_2, \\ & < x_3, y >= a_{31}x_1 + a_{32}x_2, \mathrm{rank} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} = 2. \end{aligned}$$

b. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}y$, then we have $< L_{\bar{1}}, L_{\bar{1}} >= 0$, $< x_1, y >= b_1 x_1, < x_2, y >= b_2 x_1, < x_3, y >= b_3 x_1$, $< x_1, x_1 >= a_{11}y, < x_1, x_2 >= a_{12}y$, $< x_1, x_3 >= a_{13}y, < x_2, x_2 >= a_{22}y, < x_2, x_3 >= a_{23}y, < x_3, x_3 >= a_{33}y$. It's obvious that $a_{11} = a_{22} = a_{33} = 0$. By Jacobi identities, we have $a_{12}b_1 = 0, a_{13}b_1 = 0, a_{23}b_1 = 0$. then we have either $a_{12} = a_{13} = a_{23} = 0$ or $b_1 = 0$. Since $\dim < L, L >= 2$, we have $b_1 = 0$, hence $a_{11} = a_{22} = a_{33} = b_1 = 0$. Then we have

$$L_4^5 \ : \ < x_1, y >= 0, < x_2, y >= b_2 x_1, < x_3, y >= b_3 x_1,$$

$$< x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y, < x_1, x_3 >= a_{13}y,$$
$$< x_2, x_2 >= 0, < x_2, x_3 >= a_{23}y, < x_3, x_3 >= 0,$$
$$< L_{\bar{1}}, L_{\bar{1}} >= 0; a_{12}^2 + a_{13}^2 + a_{23}^2 \neq 0, b_2^2 + b_3^2 \neq 0.$$

*D.* $\dim < L, L >= 3$.

a. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2 + \mathrm{F}x_3$, we have

$$\begin{aligned} L_4^6 \ : \ & < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, \\ & < x_1, y >= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ & < x_2, y >= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ & < x_3, y >= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \\ & \mathrm{rank}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 3. \end{aligned}$$

b. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2 + \mathrm{F}y$, then we have $< L_{\bar{1}}, L_{\bar{1}} >= 0$, $< x_1, x_1 >= b_{11}y, < x_1, x_2 >= b_{12}y$, $< x_1, x_3 >= b_{13}y, < x_2, x_2 >= b_{22}y, < x_2, x_3 >= b_{23}y, < x_3, x_3 >= b_{33}y, < x_1, y >= a_{11}x_1 + a_{12}x_2, < x_2, y >= a_{21}x_1 + a_{22}x_2$, $< x_3, y >= a_{31}x_1 + a_{32}x_2$, and

$$\mathrm{rank}\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} = 2.$$

It's clear that $b_{11} = b_{22} = b_{33} = 0$, thus we have

$$\begin{aligned} L_4^7 \ : \ & < x_1, x_1 >= 0, < x_1, x_2 >= b_{12}y, < x_1, x_3 >= b_{13}y, \\ & < x_2, x_2 >= 0, < x_2, x_3 >= b_{23}y, < x_3, x_3 >= 0, \\ & < L_{\bar{1}}, L_{\bar{1}} >= 0, b_{12}^2 + b_{13}^2 + b_{23}^2 \neq 0. \\ & < x_1, y >= a_{11}x_1 + a_{12}x_2, < x_2, y >= a_{21}x_1 + a_{22}x_2, \\ & < x_3, y >= a_{31}x_1 + a_{32}x_2, \\ & \mathrm{rank}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = 2. \end{aligned}$$

*E.* $\dim < L, L >= 4$, $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2 + \mathrm{F}x_3 + \mathrm{F}y$. Then we have $< L_{\bar{1}}, L_{\bar{1}} >= 0$, $< x_1, x_1 >= b_{11}y, < x_1, x_2 >= b_{12}y$, $< x_1, x_3 >= b_{13}y$, $< x_2, x_2 >= b_{22}y, < x_2, x_3 >= b_{23}y, < x_3, x_3 >= b_{33}y$, $< x_1, y >= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, < x_2, y >= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, < x_3, y >= a_{31}x_1 + a_{32}x_2 + a_{33}x_3$, and

$$\mathrm{rank}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 3.$$

It's obvious $b_{11} = b_{22} = b_{33} = 0$, thus we have

$$\begin{aligned} L_4^8 \ : \ & < x_1, x_1 >= 0, < x_1, x_2 >= b_{12}y, < x_1, x_3 >= b_{13}y, \\ & < x_2, x_2 >= 0, < x_2, x_3 >= b_{23}y, < x_3, x_3 >= 0, \\ & < L_{\bar{1}}, L_{\bar{1}} >= 0, b_{12}^2 + b_{13}^2 + b_{23}^2 \neq 0. \\ & < x_1, y >= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ & < x_2, y >= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ & < x_3, y >= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \\ & \mathrm{rank}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 3. \end{aligned}$$

*(3)* Assume that $\dim L_{\bar{0}} = 2, \dim L_{\bar{1}} = 2$. Then there is a basis $\{x_1, x_2, y_1, y_2\}$ in $L$, where $x_1, x_2 \in L_{\bar{0}}$ and $y_1, y_2 \in L_{\bar{1}}$. We classify $L$ by the dimension of $< L, L >$ as the following cases.

*A.* $\dim < L, L >= 0$. Then $< L, L >= 0$.

*B.* $\dim < L, L >= 1$.

a. If $< L, L >= \mathrm{F}x_1$, then we have $< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0$.

Suppose that $< x_1, y_1 >= a_{11}x_1, < x_1, y_2 >= a_{12}x_1, < x_2, y_1 >= a_{21}x_1$, $< x_2, y_2 >= a_{22}x_1$. Without loss of generality we let $a_{11} \neq 0$, then the replacements of $y_1$ by $a_{11}^{-1}y_1$, of $k$ by $\frac{a_{21}}{a_{11}}$ give $< x_1, y_1 >= x_1, < x_2, y_1 >=$

$kx_1$, thus we have

$$\begin{aligned} L_4^9 \ : \ & < x_1, y_1 >= x_1, < x_1, y_2 >= a_{12}x_1, < x_2, y_1 >= kx_1, \\ & < x_2, y_2 >= a_{22}x_1, < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0. \end{aligned}$$

b. If $< L, L >= Fy_1$, then we have $< L_{\bar{0}}, L_{\bar{1}} >= 0$.

Suppose that $< x_1, x_1 >= a_{11}y_1, < x_1, x_2 >= a_{12}y_1, < x_2, x_2 >= a_{22}y_1$. $< y_1, y_1 >= b_{11}y_1, < y_1, y_2 >= b_{12}y_1, < y_2, y_2 >= b_{22}y_1$. By definition of 2-Lie superalgebras, we have $a_{11} = a_{22} = 0$. By Jacobi identities, we have $b_{11} = 0$. Without loss of generality we may suppose that $a_{12} \neq 0$, then the replacements of $x_2$ by $a_{12}^{-1}x_2$, gives $< x_1, x_2 >= y_1$. Thus we have

$$\begin{aligned} L_4^{10} \ : \ & < x_1, x_1 >= 0, < x_1, x_2 >= y_1, < x_2, x_2 >= 0, \\ & < y_1, y_1 >= 0, < y_1, y_2 >= b_{12}y_1, < y_2, y_2 >= b_{22}y_1, \\ & < L_{\bar{0}}, L_{\bar{1}} >= 0. \end{aligned}$$

*C.* $\dim < L, L >= 2$.

a. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2$, then we have

$$\begin{aligned} L_4^{11} \ : \ & < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0, \\ & < x_1, y_1 >= a_{11}^1 x_1 + a_{11}^2 x_2, \\ & < x_1, y_2 >= a_{12}^1 x_1 + a_{12}^2 x_2, \\ & < x_2, y_1 >= a_{21}^1 x_1 + a_{21}^2 x_2, \\ & < x_2, y_2 >= a_{22}^1 x_1 + a_{22}^2 x_2, \\ & \mathrm{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{21}^1 & a_{22}^1 \\ a_{11}^2 & a_{12}^2 & a_{21}^2 & a_{22}^2 \end{pmatrix} = 2. \end{aligned}$$

b. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}y_1$, then we have $< x_1, x_1 >= a_{11}y_1, < x_1, x_2 >= a_{12}y_1, < x_2, x_2 >= a_{22}y_1, < x_1, y_1 >= b_{11}x_1, < x_1, y_2 >= b_{12}x_1, < x_2, y_1 >=$

$b_{21}x_1, < x_2, y_2 >= b_{22}x_1, < y_1, y_1 >= c_{11}y_1, < y_1, y_2 >= c_{12}y_1, < y_2, y_2 >= c_{22}y_1$. It's obvious that $a_{11} = a_{22} = 0$. By Jacobi identities, we have

$$c_{11} = 0, c_{12}c_{22} = 0, b_{11}c_{22} = 0, b_{21}c_{22} = 0, a_{12}b_{11} = 0,$$

$$c_{12}b_{11} = 0, a_{12}(b_{12} - c_{12}) = 0, b_{21}(b_{12} - c_{12}) + b_{11}b_{22} = 0.$$

Then $c_{12}c_{22} = 0, b_{11}c_{22} = 0, b_{21}c_{22} = 0, a_{12}b_{11} = 0, c_{12}b_{11} = 0$, hence $c_{22} = b_{11} = 0$ or $c_{12} = a_{12} = b_{11} = b_{21} = 0$.
Case 1. If $c_{12} = a_{12} = b_{11} = b_{21} = 0$, then $c_{22} \neq 0$, then the replacement of $y_1$ by $c_{22}y_1$ gives $< y_2, y_2 >= y_1$. Without loss of generality we may suppose that $b_{22} \neq 0$, then the replacement of $x_2$ by $b_{22}^{-1}x_2$ gives $< x_2, y_2 >= x_1$. Thus we have

$$\begin{aligned} L_4^{12} \quad : \quad & < x_1, y_1 >= 0, < x_1, y_2 >= b_{12}x_1, < x_2, y_1 >= 0, \\ & < x_2, y_2 >= x_1, < y_1, y_1 >= 0, < y_1, y_2 >= c_{12}y_1, \\ & < y_2, y_2 >= y_1, < L_{\bar{0}}, L_{\bar{0}} >= 0. \end{aligned}$$

Case 2. If $c_{22} = b_{11} = 0$, then we have $a_{12}(b_{12} - c_{12}) = 0, b_{21}(b_{12} - c_{12}) + b_{11}b_{22} = 0$, hence $b_{12} = c_{12}$ or $b_{12} \neq c_{12}, a_{12} = 0, b_{21} = 0$.
*I.* If $b_{12} = c_{12}$, then $< x_1, y_2 >= b_{12}x_1, < y_1, y_2 >= b_{12}y_1$. Thus we obtain the type

$$\begin{aligned} L_4^{13} \quad : \quad & < x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y_1, < x_2, x_2 >= 0, \\ & < x_1, y_1 >= 0, < x_1, y_2 >= b_{12}x_1, < x_2, y_1 >= b_{21}x_1, \\ & < x_2, y_2 >= b_{22}x_1, < y_1, y_1 >= 0, < y_1, y_2 >= b_{12}y_1, \\ & < y_2, y_2 >= 0; a_{12}^2 + b_{12}^2 \neq 0, b_{12}^2 + b_{21}^2 + b_{22}^2 \neq 0. \end{aligned}$$

*II.* If $b_{12} \neq c_{12}$, then $a_{12} = 0$, $b_{21} = 0$, hence $c_{12} \neq 0$. Then the replacements of $y_2$ by $c_{12}^{-1}y_2$, of $k_5$ by $\frac{b_{12}}{c_{12}}$, and of $k_6$ by $\frac{b_{22}}{c_{12}}$ give $< x_1, y_2 >= k_5x_1, <$

$x_2, y_2 >= k_6 x_1$ and $< y_1, y_2 >= y_1$. Thus we obtain the type

$$\begin{aligned} L_4^{14} \ : \ & < x_1, y_1 >= 0, < x_1, y_2 >= k_5 x_1, < x_2, y_1 >= 0, \\ & < x_2, y_2 >= k_6 x_1, < y_1, y_1 >= 0, < y_1, y_2 >= y_1, \\ & < y_2, y_2 >= 0, < L_{\bar{0}}, L_{\bar{0}} >= 0; \ k_5^2 + k_6^2 \neq 0. \end{aligned}$$

c. If $< L, L >= \mathrm{F}y_1 + \mathrm{F}y_2$, then it's clear that $< L_{\bar{0}}, L_{\bar{1}} >= 0$. Suppose that $< x_1, x_1 >= a_{11}^1 y_1 + a_{11}^2 y_2$, $< x_1, x_2 >= a_{12}^1 y_1 + a_{12}^2 y_2$, $< x_2, x_2 >= a_{22}^1 y_1 + a_{22}^2 y_2$, $< y_1, y_1 >= b_{11}^1 y_1 + b_{11}^2 y_2$, $< y_1, y_2 >= b_{12}^1 y_1 + b_{12}^2 y_2$, $< y_2, y_2 >= b_{22}^1 y_1 + b_{22}^2 y_2$, and

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{22}^1 & b_{11}^1 & b_{12}^1 & b_{22}^1 \\ a_{11}^2 & a_{12}^2 & a_{22}^2 & b_{11}^2 & b_{12}^2 & b_{22}^2 \end{pmatrix} = 2.$$

By the definition of 2-Lie superalgebras, we have $a_{11}^1 = a_{11}^2 = a_{22}^1 = a_{22}^2 = 0$. By Jacobi identities, we have

$$\begin{aligned} & b_{11}^1 b_{11}^1 + b_{11}^2 b_{12}^1 = 0, \quad b_{11}^1 b_{12}^1 + b_{11}^2 b_{22}^1 = 0; \\ & b_{11}^1 b_{11}^2 + b_{11}^2 b_{12}^2 = 0, \quad b_{11}^1 b_{12}^2 + b_{11}^2 b_{22}^2 = 0; \\ & b_{12}^1 b_{22}^1 + b_{22}^1 b_{22}^2 = 0, \quad b_{11}^1 b_{22}^1 + b_{12}^1 b_{22}^2 = 0; \\ & b_{22}^2 b_{22}^2 + b_{12}^2 b_{22}^1 = 0, \quad b_{11}^2 b_{22}^1 + b_{12}^2 b_{22}^2 = 0; \\ & a_{12}^1 b_{11}^1 + a_{12}^2 b_{12}^1 = 0, \quad a_{12}^1 b_{11}^2 + a_{12}^2 b_{12}^2 = 0; \\ & a_{12}^1 b_{12}^1 + a_{12}^2 b_{22}^1 = 0, \quad a_{12}^1 b_{12}^2 + a_{12}^2 b_{22}^2 = 0. \end{aligned}$$

If $a_{12}^1 = 0, a_{12}^2 \neq 0$, then we have $b_{12}^1 = b_{12}^2 = b_{22}^1 = b_{22}^2 = b_{11}^1 = 0$, hence $r = 1$. If $a_{12}^1 = a_{12}^2 = 0$, we have $\frac{b_{11}^2}{b_{11}^1} = \frac{b_{12}^2}{b_{12}^1} = \frac{b_{22}^2}{b_{22}^1}$, then $r = 1$. If $a_{12}^1 \neq 0, a_{12}^2 = 0$, we have $b_{11}^2 = b_{12}^2 = b_{22}^2 = b_{11}^1 = 0$, hence $r = 1$. If $a_{12}^1 a_{12}^2 \neq 0$, we have $\frac{b_{11}^2}{b_{11}^1} = \frac{b_{12}^2}{b_{12}^1} = \frac{b_{22}^2}{b_{22}^1} = -\frac{b_{11}^1}{b_{12}^1}$. Note that $a_{12}^1 b_{11}^1 + a_{12}^2 b_{12}^1 = 0$, then $\frac{a_{12}^2}{a_{12}^1} = -\frac{b_{11}^1}{b_{12}^1}$, hence $\frac{b_{11}^2}{b_{11}^1} = \frac{b_{12}^2}{b_{12}^1} = \frac{b_{22}^2}{b_{22}^1} = -\frac{b_{11}^1}{b_{12}^1} = \frac{a_{12}^2}{a_{12}^1}$, that is, $r = 1$. In

conclusion $r = 1$. If $r = 1$, then $\dim < L, L >= 1$, a contradiction. So we claim that there is no 2-Lie superalgebra in this case.

*D.* $\dim < L, L >= 3$.

a. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2 + \mathrm{F}y_1$, then we have $< x_1, x_1 >= a_{11}y_1$, $< x_1, x_2 >= a_{12}y_1, < x_2, x_2 >= a_{22}y_1$, $< y_1, y_1 >= b_{11}y_1, < y_1, y_2 >= b_{12}y_1, < y_2, y_2 >= b_{22}y_1$, $< x_1, y_1 >= a^1_{11}x_1 + a^2_{11}x_2, < x_1, y_2 >= a^1_{12}x_1 + a^2_{12}x_2$, $< x_2, y_1 >= a^1_{21}x_1 + a^2_{21}x_2, < x_2, y_2 >= a^1_{22}x_1 + a^2_{22}x_2$, and

$$\mathrm{rank}\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{21} & a^1_{22} \\ a^2_{11} & a^2_{12} & a^2_{21} & a^2_{22} \end{pmatrix} = 2.$$

By Jacobi identities, we have $b_{11} = 0, b_{22}b_{12} = 0, b_{22}a^1_{11} = 0$, $b_{22}a^2_{11} = 0, b_{22}a^1_{21} = 0, b_{22}a^2_{21} = 0$. Then we have $b_{11} = b_{22} = 0$ or $b_{11} = b_{12} = a^1_{11} = a^2_{11} = a^1_{21} = a^2_{21} = 0$.

*Case 1.* If $b_{11} = b_{22} = 0$, then we have

$$\begin{aligned} L_4^{15} \ : \ & < x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y_1 \\ & < x_2, x_2 >= 0, < y_1, y_1 >= 0, < y_1, y_2 >= b_{12}y_1, \\ & < y_2, y_2 >= 0, a^2_{12} + b^2_{12} \neq 0. \\ & < x_1, y_1 >= a^1_{11}x_1 + a^2_{11}x_2, < x_1, y_2 >= a^1_{12}x_1 + a^2_{12}x_2, \\ & < x_2, y_1 >= a^1_{21}x_1 + a^2_{21}x_2, < x_2, y_2 >= a^1_{22}x_1 + a^2_{22}x_2, \\ & \mathrm{rank}\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{21} & a^1_{22} \\ a^2_{11} & a^2_{12} & a^2_{21} & a^2_{22} \end{pmatrix} = 2. \end{aligned}$$

*Case 2.* If $b_{11} = b_{12} = a^1_{11} = a^2_{11} = a^1_{21} = a^2_{21} = 0$, then we obtain the type

$$\begin{aligned} L_4^{16} \ : \ & < x_1, x_1 >= 0, < x_1, x_2 >= a_{12}y_1, < x_2, x_2 >= 0, \\ & < y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_2, y_2 >= b_{22}y_1, a^2_{12} + b^2_{22} \neq 0. \\ & < x_1, y_1 >= 0, < x_1, y_2 >= a^1_{12}x_1 + a^2_{12}x_2, \\ & < x_2, y_1 >= 0, < x_2, y_2 >= a^1_{22}x_1 + a^2_{22}x_2, \\ & \mathrm{rank}\begin{pmatrix} 0 & a^1_{12} & 0 & a^1_{22} \\ 0 & a^2_{12} & 0 & a^2_{22} \end{pmatrix} = 2. \end{aligned}$$

b. If $< L, L >= \mathrm{F}x_1 + \mathrm{F}y_1 + \mathrm{F}y_2$, then we have $< x_1, x_1 >= a_{11}^1 y_1 + a_{11}^2 y_2$, $< x_1, x_2 >= a_{12}^1 y_1 + a_{12}^2 y_2$, $< x_2, x_2 >= a_{22}^1 y_1 + a_{22}^2 y_2$, $< y_1, y_1 >= b_{11}^1 y_1 + b_{11}^2 y_2$, $< y_1, y_2 >= b_{12}^1 y_1 + b_{12}^2 y_2$, $< y_2, y_2 >= b_{22}^1 y_1 + b_{22}^2 y_2$, $< x_1, y_1 >= c_{11} x_1$, $< x_1, y_2 >= c_{12} x_1$, $< x_2, y_1 >= c_{21} x_1$, $< x_2, y_2 >= c_{22} x_1$,

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{22}^1 & b_{11}^1 & b_{12}^1 & b_{22}^1 \\ a_{11}^2 & a_{12}^2 & a_{22}^2 & b_{11}^2 & b_{12}^2 & b_{22}^2 \end{pmatrix} = r.$$

Clearly, we have $a_{11}^1 = a_{11}^2 = a_{22}^1 = a_{22}^2 = 0$.

By Jacobi identities, we have

$$\begin{aligned}
&b_{11}^1 b_{11}^1 + b_{11}^2 b_{12}^1 = 0, \quad b_{11}^1 b_{12}^1 + b_{11}^2 b_{22}^1 = 0, \\
&b_{11}^1 b_{11}^2 + b_{11}^2 b_{12}^2 = 0, \quad b_{11}^1 b_{12}^2 + b_{11}^2 b_{22}^2 = 0, \\
&b_{12}^1 b_{22}^1 + b_{22}^1 b_{22}^2 = 0, \quad b_{11}^1 b_{22}^1 + b_{12}^1 b_{22}^2 = 0, \\
&b_{22}^2 b_{22}^2 + b_{12}^2 b_{22}^1 = 0, \quad b_{11}^2 b_{22}^1 + b_{12}^2 b_{22}^2 = 0, \\
&a_{12}^1 b_{11}^1 + a_{12}^2 b_{12}^1 = 0, \quad a_{12}^1 b_{11}^2 + a_{12}^2 b_{12}^2 = 0, \\
&a_{12}^1 b_{12}^1 + a_{12}^2 b_{22}^1 = 0, \quad a_{12}^1 b_{12}^2 + a_{12}^2 b_{22}^2 = 0, \\
&c_{11} b_{11}^1 + c_{12} b_{11}^2 = 0, \quad c_{11} b_{22}^1 + c_{12} b_{22}^2 = 0, \\
&c_{21} b_{11}^1 + c_{22} b_{11}^2 = 0, \quad c_{21} b_{22}^1 + c_{22} b_{22}^2 = 0.
\end{aligned}$$

From the proof before, we know $r = 1$. If $r = 1$, this contradicts $\dim < L, L >= 2$. So we claim that there is no 2-Lie superalgebra in this case.

*E.* $\dim < L, L >= 4$, $< L, L >= \mathrm{F}x_1 + \mathrm{F}x_2 + \mathrm{F}y_1 + \mathrm{F}y_2$. Then we have $< x_1, x_1 >= a_{11}^1 y_1 + a_{11}^2 y_2$, $< x_1, x_2 >= a_{12}^1 y_1 + a_{12}^2 y_2$, $< x_2, x_2 >= a_{22}^1 y_1 + a_{22}^2 y_2$, $< y_1, y_1 >= b_{11}^1 y_1 + b_{11}^2 y_2$, $< y_1, y_2 >= b_{12}^1 y_1 + b_{12}^2 y_2$, $< y_2, y_2 >=$

$b^1_{22}y_1 + b^2_{22}y_2, < x_1, y_1 >= c^1_{11}x_1 + c^2_{11}x_2, < x_1, y_2 >= c^1_{12}x_1 + c^2_{12}x_2, < x_2, y_1 >= c^1_{21}x_1 + c^2_{21}x_2, < x_2, y_2 >= c^1_{22}x_1 + c^2_{22}x_2,$

$$\text{rank}\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{22} & b^1_{11} & b^1_{12} & b^1_{22} \\ a^2_{11} & a^2_{12} & a^2_{22} & b^2_{11} & b^2_{12} & b^2_{22} \end{pmatrix} = r_1, \text{rank}\begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{21} & c^1_{22} \\ c^2_{11} & c^2_{12} & c^2_{21} & c^2_{22} \end{pmatrix} = 2.$$

From the proof before, we know $r_1 = 1$, which is a contradiction. So we claim that there is no 2-Lie superalgebra in this case.

*(4)* Assume that $\dim L_{\bar{0}} = 1$ and $\dim L_{\bar{1}} = 3$. Then there is a basis $\{x, y_1, y_2, y_3\}$ in $L$, where $x \in L_{\bar{0}}$ and $y_1, y_2, y_3 \in L_{\bar{1}}$. We classify $L$ by the dimension of $< L, L >$ as the following cases.

*A.* $\dim < L, L >= 0$. Then $< L, L >= 0$.

*B.* $\dim < L, L >= 1$.

a. If $< L, L >= \mathrm{F}x$, then we have $< L_{\bar{0}}, L_{\bar{0}} >= 0$, $< L_{\bar{1}}, L_{\bar{1}} >= 0$, $< x, y_1 >= a_1x, < x, y_2 >= a_2x, < x, y_3 >= a_3x$. Without loss of generality we let $a_1 \neq 0$, then the replacements of $y_{11}$ by $a_1^{-1}y_1$, of $y_{12}$ by $y_2 - \frac{a_2}{a_1}y_1$, and of $y_{13}$ by $y_3 - \frac{a_3}{a_1}y_1$, give $< x, y_{11} >= x, < x, y_{12} >= 0$ and $< x, y_{13} >= 0$, thus we have

$$\begin{aligned} L_4^{17} \quad : \quad & < x, y_1 >= x, < x, y_2 >= 0, < x, y_3 >= 0, \\ & < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{1}}, L_{\bar{1}} >= 0. \end{aligned}$$

b. If $< L, L >= \mathrm{F}y_1$, then it's clear $< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0$. Suppose that $< y_1, y_1 >= b_{11}y_1, < y_1, y_2 >= b_{12}y_1, < y_1, y_3 >= b_{13}y_1, < y_2, y_2 >= b_{22}y_1, < y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= b_{33}y_1$. By Jacobi identities, we have

$$b_{11} = 0, b_{12}b_{22} = 0, b_{12}b_{33} = 0, b_{13}b_{22} = 0, b_{13}b_{33} = 0.$$

Thus we have either $b_{22} = b_{33} = 0$ or $b_{12} = b_{13} = 0$.

*Case 1.* If $b_{22} = b_{33} = 0$, then we have

$$L_4^{18} \quad : \quad < y_1, y_1 >= 0, < y_1, y_2 >= b_{12}y_1, < y_1, y_3 >= b_{13}y_1,$$

$$< y_2, y_2 >= 0, < y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= 0,$$
$$< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0; b_{12}^2 + b_{13}^2 + b_{23}^2 \neq 0.$$

*Case 2.* If $b_{12} = b_{13} = 0$, then we obtain the type

$$\begin{aligned} L_4^{19} \quad : \quad & < y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_1, y_3 >= 0, \\ & < y_2, y_2 >= b_{22}y_1, < y_2, y_3 >= b_{23}y_1, < y_3, y_3 >= b_{33}y_1, \\ & < L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0; b_{22}^2 + b_{23}^2 + b_{33}^2 \neq 0. \end{aligned}$$

*C.* $\dim < L, L >= 2$.

a. If $< L, L >= \mathrm{F}y_1 + \mathrm{F}y_2$, then it's clear that $< L_{\bar{0}}, L_{\bar{0}} >= 0, < L_{\bar{0}}, L_{\bar{1}} >= 0$. Suppose that $< y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2, < y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2, < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2, < y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2, < y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2$,

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2 \end{pmatrix} = r.$$

If $r = 1$, then $\dim < L, L >= 1$, which contradicts $\dim < L, L >= 2$. Hence $r = 2$. By Jacobi identities, we have

$$a_{11}^1 a_{11}^1 + a_{11}^2 a_{12}^1 = 0, \quad a_{11}^2 a_{33}^1 + a_{12}^2 a_{33}^2 = 0;$$

$$a_{11}^2 a_{11}^1 + a_{11}^2 a_{12}^2 = 0, \quad a_{22}^1 a_{12}^1 + a_{22}^1 a_{22}^2 = 0;$$
$$a_{11}^1 a_{12}^1 + a_{11}^2 a_{22}^1 = 0, \quad a_{22}^2 a_{22}^2 + a_{22}^1 a_{12}^2 = 0;$$
$$a_{11}^1 a_{12}^2 + a_{11}^2 a_{22}^2 = 0, \quad a_{13}^1 a_{22}^1 + a_{23}^1 a_{22}^2 = 0;$$
$$a_{11}^1 a_{13}^1 + a_{11}^2 a_{23}^1 = 0, \quad a_{13}^2 a_{22}^1 + a_{23}^2 a_{22}^2 = 0;$$
$$a_{11}^1 a_{13}^2 + a_{11}^2 a_{23}^2 = 0, \quad a_{12}^1 a_{33}^1 + a_{22}^1 a_{33}^2 = 0;$$
$$a_{11}^1 a_{22}^1 + a_{12}^1 a_{22}^2 = 0, \quad a_{12}^2 a_{33}^1 + a_{22}^2 a_{33}^2 = 0;$$

$$a_{11}^2a_{22}^1+a_{12}^2a_{22}^2=0,\quad a_{13}^1a_{33}^1+a_{23}^1a_{33}^2=0;$$
$$a_{11}^1a_{33}^1+a_{12}^1a_{33}^2=0,\quad a_{13}^2a_{33}^1+a_{23}^2a_{33}^2=0.$$

Thus we obtain

$$\begin{aligned}L_4^{20}\ :\ &< y_1,y_1>=a_{11}^1y_1+a_{11}^2y_2, < y_1,y_2>=a_{12}^1y_1+a_{12}^2y_2,\\ &< y_1,y_3>=a_{13}^1y_1+a_{13}^2y_2, < y_2,y_2>=a_{22}^1y_1+a_{22}^2y_2,\\ &< y_2,y_3>=a_{23}^1y_1+a_{23}^2y_2, < y_3,y_3>=a_{33}^1y_1+a_{33}^2y_2,\\ &< L_{\bar 0},L_{\bar 0}>=0, < L_{\bar 0},L_{\bar 1}>=0,\\ &\operatorname{rank}\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1\\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2\end{pmatrix}=2.\end{aligned}$$

b. If $<L,L>=\mathrm{F}x+\mathrm{F}y_1$, then it's obvious that $<L_{\bar 0},L_{\bar 0}>=0$. Suppose that $<x,y_1>=a_1x, <x,y_2>=a_2x, <x,y_3>=a_3x$, $<y_1,y_1>=b_{11}y_1, <y_1,y_2>=b_{12}y_1$, $<y_1,y_3>=b_{13}y_1$, $<y_2,y_2>=b_{22}y_1$, $<y_2,y_3>=b_{23}y_1, <y_3,y_3>=b_{33}y_1$. By Jacobi identities, we have

$$b_{11}=0, a_1b_{12}=0, a_1b_{13}=0, a_1b_{22}=0, a_1b_{23}=0,$$
$$a_1b_{33}=0, b_{12}b_{22}=0, b_{12}b_{33}=0, b_{13}b_{22}=0, b_{13}b_{33}=0.$$

Then we have either $a_1=b_{12}=b_{13}=0$ or $a_1=b_{22}=b_{33}=0$.

*I.* If $a_1=b_{12}=b_{13}=0$, then we obtain

$$\begin{aligned}L_4^{21}\ :\ &<x,y_1>=0, <x,y_2>=a_2x, <x,y_3>=a_3x, <y_1,y_1>=0,\\ &<y_1,y_2>=0, <y_1,y_3>=0, <y_2,y_2>=b_{22}y_1, <y_2,y_3>=b_{23}y_1,\\ &<y_3,y_3>=b_{33}y_1, <L_{\bar 0},L_{\bar 0}>=0, b_{22}^2+b_{23}^2+b_{33}^2\neq 0, a_2^2+a_3^2\neq 0.\end{aligned}$$

*II.* If $a_1=b_{22}=b_{33}=0$, then we have

$$\begin{aligned}L_4^{22}\ :\ &<x,y_1>=0, <x,y_2>=a_2x, <x,y_3>=a_3x,\\ &<y_1,y_1>=0, <y_1,y_2>=b_{12}y_1, <y_1,y_3>=b_{13}y_1, <y_2,y_2>=0,\\ &<y_2,y_3>=b_{23}y_1, <y_3,y_3>=0, <L_{\bar 0},L_{\bar 0}>=0;\\ &b_{12}^2+b_{13}^2+b_{23}^2\neq 0, a_2^2+a_3^2\neq 0.\end{aligned}$$

*D.* $\dim < L, L >= 3$,

a. If $< L, L >= \mathrm{F}y_1 + \mathrm{F}y_2 + \mathrm{F}y_3$, then $< y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2 + a_{11}^3 y_3$, $< y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2 + a_{12}^3 y_3$, $< y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2 + a_{13}^3 y_3$, $< y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2 + a_{22}^3 y_3$, $< y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2 + a_{23}^3 y_3$, $< y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2 + a_{33}^3 y_3$, and

$$\operatorname{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2 \\ a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{22}^3 & a_{23}^3 & a_{33}^3 \end{pmatrix} = r.$$

If $r = 1$, then $\dim < L, L >= 1$, which contradicts $\dim < L, L >= 2$. If $r = 2$, then $\dim < L, L >= 2$, and this contradicts $\dim < L, L >= 3$. Then $r = 3$. By Jacobi identities, we have

$$a_{11}^1 a_{11}^1 + a_{11}^2 a_{12}^1 + a_{11}^3 a_{13}^1 = 0, \quad a_{12}^1 a_{22}^1 + a_{22}^1 a_{22}^2 + a_{22}^3 a_{23}^1 = 0;$$

$$a_{11}^1 a_{11}^2 + a_{11}^2 a_{12}^2 + a_{11}^3 a_{13}^2 = 0, \quad a_{22}^1 a_{12}^2 + a_{22}^2 a_{22}^2 + a_{22}^3 a_{23}^2 = 0;$$

$$a_{11}^1 a_{11}^3 + a_{11}^2 a_{12}^3 + a_{11}^3 a_{13}^3 = 0, \quad a_{12}^3 a_{22}^1 + a_{22}^2 a_{22}^3 + a_{22}^3 a_{23}^3 = 0;$$

$$a_{11}^1 a_{12}^1 + a_{11}^2 a_{22}^1 + a_{11}^3 a_{23}^1 = 0, \quad a_{13}^1 a_{33}^1 + a_{23}^1 a_{33}^2 + a_{33}^1 a_{33}^3 = 0;$$

$$a_{11}^1 a_{12}^2 + a_{11}^2 a_{22}^2 + a_{11}^3 a_{23}^2 = 0, \quad a_{13}^2 a_{33}^1 + a_{23}^2 a_{33}^2 + a_{33}^2 a_{33}^3 = 0;$$

$$a_{11}^1 a_{12}^3 + a_{11}^2 a_{22}^3 + a_{11}^3 a_{23}^3 = 0, \quad a_{13}^3 a_{33}^1 + a_{23}^3 a_{33}^2 + a_{33}^3 a_{33}^3 = 0;$$

$$a_{11}^1 a_{13}^1 + a_{11}^2 a_{23}^1 + a_{11}^3 a_{33}^1 = 0, \quad a_{22}^1 a_{11}^1 + a_{22}^2 a_{12}^1 + a_{22}^3 a_{13}^1 = 0;$$

$$a_{11}^1 a_{13}^2 + a_{11}^2 a_{23}^2 + a_{11}^3 a_{33}^2 = 0, \quad a_{22}^1 a_{11}^2 + a_{22}^2 a_{12}^2 + a_{22}^3 a_{13}^2 = 0;$$

$$a_{11}^1 a_{13}^3 + a_{11}^2 a_{23}^3 + a_{11}^3 a_{33}^3 = 0, \quad a_{22}^1 a_{11}^3 + a_{22}^2 a_{12}^3 + a_{22}^3 a_{13}^3 = 0;$$

$$a_{22}^1 a_{13}^1 + a_{22}^2 a_{23}^1 + a_{22}^3 a_{33}^1 = 0, \quad a_{33}^1 a_{11}^1 + a_{33}^2 a_{12}^1 + a_{33}^3 a_{13}^1 = 0;$$

$$a_{22}^1 a_{13}^2 + a_{22}^2 a_{23}^2 + a_{22}^3 a_{33}^2 = 0, \quad a_{33}^1 a_{11}^2 + a_{33}^2 a_{12}^2 + a_{33}^3 a_{13}^2 = 0;$$

$$a_{22}^1 a_{13}^3 + a_{22}^2 a_{23}^3 + a_{22}^3 a_{33}^3 = 0, \quad a_{33}^1 a_{11}^3 + a_{33}^2 a_{12}^3 + a_{33}^3 a_{13}^3 = 0;$$

$$a_{33}^1a_{13}^1+a_{33}^2a_{23}^1+a_{33}^3a_{33}^1=0,\quad a_{33}^1a_{13}^2+a_{33}^2a_{23}^2+a_{33}^3a_{33}^2=0;$$

$$a_{33}^1a_{13}^3+a_{33}^2a_{23}^3+a_{33}^3a_{33}^3=0,$$

$$(a_{11}^1+a_{12}^1-a_{13}^3)a_{23}^1+(a_{13}^1-a_{23}^2)a_{12}^1-a_{13}^2a_{22}^1+a_{12}^3a_{33}^1=0.$$

Thus we obtain

$$\begin{aligned}L_4^{23}\ :\ &< y_1,y_1>=a_{11}^1y_1+a_{11}^2y_2+a_{11}^3y_3, < y_1,y_2>=a_{12}^1y_1+a_{12}^2y_2+a_{12}^3y_3,\\ &< y_1,y_3>=a_{13}^1y_1+a_{13}^2y_2+a_{13}^3y_3, < y_2,y_2>=a_{22}^1y_1+a_{22}^2y_2+a_{22}^3y_3,\\ &< y_2,y_3>=a_{23}^1y_1+a_{23}^2y_2+a_{23}^3y_3, < y_3,y_3>=a_{33}^1y_1+a_{33}^2y_2+a_{33}^3y_3,\\ &\operatorname{rank}\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1\\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2\\ a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{22}^3 & a_{23}^3 & a_{33}^3\end{pmatrix}=3.\end{aligned}$$

b. If $<L,L>=\mathrm{F}x+\mathrm{F}y_1+\mathrm{F}y_2$, then it's clear $<L_{\bar{0}},L_{\bar{0}}>=0$. Then we have $<x,y_1>=a_1x, <x,y_2>=a_2x, <x,y_3>=a_3x$, $<y_1,y_1>=a_{11}^1y_1+a_{11}^2y_2, <y_1,y_2>=a_{12}^1y_1+a_{12}^2y_2$, $<y_1,y_3>=a_{13}^1y_1+a_{13}^2y_2, <y_2,y_2>=a_{22}^1y_1+a_{22}^2y_2$, $<y_2,y_3>=a_{23}^1y_1+a_{23}^2y_2, <y_3,y_3>=a_{33}^1y_1+a_{33}^2y_2$, and

$$\operatorname{rank}\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1\\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2\end{pmatrix}=r.$$

If $r=1$, then $\dim <L,L>=1$, which contradicts $\dim <L,L>=2$. Hence $r=2$. By Jacobi identities, we have

$$a_{11}^1a_{11}^1+a_{11}^2a_{12}^1=0,\quad a_{11}^2a_{33}^1+a_{12}^2a_{33}^2=0.$$

$$a_{11}^2a_{11}^1+a_{11}^2a_{12}^2=0,\quad a_{22}^1a_{12}^1+a_{22}^1a_{22}^2=0.$$

$$a_{11}^1a_{12}^1+a_{11}^2a_{22}^1=0,\quad a_{22}^2a_{22}^2+a_{22}^1a_{12}^2=0.$$

$$a_{11}^1a_{12}^2+a_{11}^2a_{22}^2=0,\quad a_{13}^1a_{22}^1+a_{23}^1a_{22}^2=0.$$

$$a_{11}^1a_{13}^1+a_{11}^2a_{23}^1=0,\quad a_{13}^2a_{22}^1+a_{23}^2a_{22}^2=0.$$

$$a^1_{11}a^2_{13}+a^2_{11}a^2_{23}=0,\quad a^1_{12}a^1_{33}+a^1_{22}a^2_{33}=0.$$
$$a^1_{11}a^1_{22}+a^1_{12}a^2_{22}=0,\quad a^2_{12}a^1_{33}+a^2_{22}a^2_{33}=0.$$
$$a^2_{11}a^1_{22}+a^2_{12}a^2_{22}=0,\quad a^1_{13}a^1_{33}+a^1_{23}a^2_{33}=0.$$
$$a^1_{11}a^1_{33}+a^1_{12}a^2_{33}=0,\quad a^2_{13}a^1_{33}+a^2_{23}a^2_{33}=0.$$
$$a_1a^1_{11}+a_2a^2_{11}=0,\quad a_1a^1_{12}+a_2a^2_{12}=0.$$
$$a_1a^1_{13}+a_2a^2_{13}=0,\quad a_1a^1_{22}+a_2a^2_{22}=0.$$
$$a_1a^1_{23}+a_2a^2_{23}=0,\quad a_1a^1_{33}+a_2a^2_{33}=0.$$

Thus we obtain

$$\begin{aligned}L_4^{24}\ :\ &<x,y_1>=a_1x,<x,y_2>=a_2x,<x,y_3>=a_3x,\\ &<y_1,y_1>=a^1_{11}y_1+a^2_{11}y_2,<y_1,y_2>=a^1_{12}y_1+a^2_{12}y_2,\\ &<y_1,y_3>=a^1_{13}y_1+a^2_{13}y_2,<y_2,y_2>=a^1_{22}y_1+a^2_{22}y_2,\\ &<y_2,y_3>=a^1_{23}y_1+a^2_{23}y_2,<y_3,y_3>=a^1_{33}y_1+a^2_{33}y_2,\\ &<L_{\bar 0},L_{\bar 0}>=0;\ a_1^2+a_2^2+a_3^2\neq 0,\\ &\operatorname{rank}\begin{pmatrix}a^1_{11}&a^1_{12}&a^1_{13}&a^1_{22}&a^1_{23}&a^1_{33}\\ a^2_{11}&a^2_{12}&a^2_{13}&a^2_{22}&a^2_{23}&a^2_{33}\end{pmatrix}=2.\end{aligned}$$

*E.* $\dim<L,L>=4$, $<L,L>=\mathrm{F}x+\mathrm{F}y_1+\mathrm{F}y_2+\mathrm{F}y_3$. Hence

$$\begin{aligned}&<x,y_1>=a_1x,<x,y_2>=a_2x,<x,y_3>=a_3x,<L_{\bar 0},L_{\bar 0}>=0.\\ &<y_1,y_1>=a^1_{11}y_1+a^2_{11}y_2+a^3_{11}y_3,<y_1,y_2>=a^1_{12}y_1+a^2_{12}y_2+a^3_{12}y_3,\\ &<y_1,y_3>=a^1_{13}y_1+a^2_{13}y_2+a^3_{13}y_3,<y_2,y_2>=a^1_{22}y_1+a^2_{22}y_2+a^3_{22}y_3,\\ &<y_2,y_3>=a^1_{23}y_1+a^2_{23}y_2+a^3_{23}y_3,<y_3,y_3>=a^1_{33}y_1+a^2_{33}y_2+a^3_{33}y_3,\end{aligned}$$

$$\operatorname{rank}\begin{pmatrix}a^1_{11}&a^1_{12}&a^1_{13}&a^1_{22}&a^1_{23}&a^1_{33}\\ a^2_{11}&a^2_{12}&a^2_{13}&a^2_{22}&a^2_{23}&a^2_{33}\\ a^3_{11}&a^3_{12}&a^3_{13}&a^3_{22}&a^3_{23}&a^3_{33}\end{pmatrix}=r.$$

If $r = 1$, then $\dim < L, L >= 1$, which contradicts $\dim < L, L >= 2$. If $r = 2$, then $\dim < L, L >= 2$, and this contradicts $\dim < L, L >= 3$. Then $r = 3$. By Jacobi identities, we have

$$a_{11}^1 a_{11}^1 + a_{11}^2 a_{12}^1 + a_{11}^3 a_{13}^1 = 0, \qquad a_{12}^1 a_{22}^1 + a_{22}^1 a_{22}^2 + a_{22}^3 a_{23}^1 = 0,$$

$$a_{11}^1 a_{11}^2 + a_{11}^2 a_{12}^2 + a_{11}^3 a_{13}^2 = 0, \qquad a_{22}^1 a_{12}^2 + a_{22}^2 a_{22}^2 + a_{22}^3 a_{23}^2 = 0,$$

$$a_{11}^1 a_{11}^3 + a_{11}^2 a_{12}^3 + a_{11}^3 a_{13}^3 = 0, \qquad a_{12}^3 a_{22}^1 + a_{22}^2 a_{22}^3 + a_{22}^3 a_{23}^3 = 0,$$

$$a_{11}^1 a_{12}^1 + a_{11}^2 a_{22}^1 + a_{11}^3 a_{23}^1 = 0, \qquad a_{13}^1 a_{33}^1 + a_{23}^1 a_{33}^2 + a_{33}^1 a_{33}^3 = 0,$$

$$a_{11}^1 a_{12}^2 + a_{11}^2 a_{22}^2 + a_{11}^3 a_{23}^2 = 0, \qquad a_{13}^2 a_{33}^1 + a_{23}^2 a_{33}^2 + a_{33}^2 a_{33}^3 = 0,$$

$$a_{11}^1 a_{12}^3 + a_{11}^2 a_{22}^3 + a_{11}^3 a_{23}^3 = 0, \qquad a_{13}^3 a_{33}^1 + a_{23}^3 a_{33}^2 + a_{33}^3 a_{33}^3 = 0,$$

$$a_{11}^1 a_{13}^1 + a_{11}^2 a_{23}^1 + a_{11}^3 a_{33}^1 = 0, \qquad a_{22}^1 a_{11}^1 + a_{22}^2 a_{12}^1 + a_{22}^3 a_{13}^1 = 0,$$

$$a_{11}^1 a_{13}^2 + a_{11}^2 a_{23}^2 + a_{11}^3 a_{33}^2 = 0, \qquad a_{22}^1 a_{11}^2 + a_{22}^2 a_{12}^2 + a_{22}^3 a_{13}^2 = 0,$$

$$a_{11}^1 a_{13}^3 + a_{11}^2 a_{23}^3 + a_{11}^3 a_{33}^3 = 0, \qquad a_{22}^1 a_{11}^3 + a_{22}^2 a_{12}^3 + a_{22}^3 a_{13}^3 = 0,$$

$$a_{22}^1 a_{13}^1 + a_{22}^2 a_{23}^1 + a_{22}^3 a_{33}^1 = 0, \qquad a_{33}^1 a_{11}^1 + a_{33}^2 a_{12}^1 + a_{33}^3 a_{13}^1 = 0,$$

$$a_{22}^1 a_{13}^2 + a_{22}^2 a_{23}^2 + a_{22}^3 a_{33}^2 = 0, \qquad a_{33}^1 a_{11}^2 + a_{33}^2 a_{12}^2 + a_{33}^3 a_{13}^2 = 0,$$

$$a_{22}^1 a_{13}^3 + a_{22}^2 a_{23}^3 + a_{22}^3 a_{33}^3 = 0, \qquad a_{33}^1 a_{11}^3 + a_{33}^2 a_{12}^3 + a_{33}^3 a_{13}^3 = 0,$$

$$a_{33}^1 a_{13}^1 + a_{33}^2 a_{23}^1 + a_{33}^3 a_{33}^1 = 0, \qquad a_{33}^1 a_{13}^2 + a_{33}^2 a_{23}^2 + a_{33}^3 a_{33}^2 = 0,$$

$$a_{33}^1 a_{13}^3 + a_{33}^2 a_{23}^3 + a_{33}^3 a_{33}^3 = 0, \qquad a_1 a_{33}^1 + a_2 a_{33}^2 + + a_3 a_{33}^3 = 0,$$

$$a_1 a_{11}^1 + a_2 a_{11}^2 + + a_3 a_{11}^3 = 0, \qquad a_1 a_{12}^1 + a_2 a_{12}^2 + + a_3 a_{12}^3 = 0,$$

$$a_1 a_{13}^1 + a_2 a_{13}^2 + + a_3 a_{13}^3 = 0, \qquad a_1 a_{22}^1 + a_2 a_{22}^2 + + a_3 a_{22}^3 = 0,$$

$$a_1 a_{23}^1 + a_2 a_{23}^2 + + a_3 a_{23}^3 = 0,$$

$$(a_{11}^1 + a_{12}^1 - a_{13}^3) a_{23}^1 + (a_{13}^1 - a_{23}^2) a_{12}^1 - a_{13}^2 a_{22}^1 + a_{12}^3 a_{33}^1 = 0.$$

Thus we obtain

$$\begin{aligned} L_4^{25} \quad : \quad & < x, y_1 >= a_1 x, < x, y_2 >= a_2 x, < x, y_3 >= a_3 x, < L_{\bar{0}}, L_{\bar{0}} >= 0, \\ & < y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2 + a_{11}^3 y_3, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2 + a_{12}^3 y_3, \\ & < y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2 + a_{13}^3 y_3, < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2 + a_{22}^3 y_3, \\ & < y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2 + a_{23}^3 y_3, < y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2 + a_{33}^3 y_3; \\ & a_1^2 + a_2^2 + a_3^2 \neq 0, \\ & \text{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{22}^1 & a_{23}^1 & a_{33}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{22}^2 & a_{23}^2 & a_{33}^2 \\ a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{22}^3 & a_{23}^3 & a_{33}^3 \end{pmatrix} = 3. \end{aligned}$$

*(5)* Assume that $\dim L_{\bar{1}} = 4$, then there is a basis $\{y_1, y_2, y_3, y_4\}$ in $L_{\bar{1}}$. We classify $L$ by the dimension of $< L, L >$ as the following cases.

*A.* If $\dim < L, L >= 0$, then $< L, L >= 0$.

*B.* If $\dim < L, L >= 1$, $< L, L >= \mathrm{F}y_1$. Suppose that $< y_1, y_1 >= b_{11} y_1$, $< y_1, y_2 >= b_{12} y_1$, $< y_1, y_3 >= b_{13} y_1$, $< y_1, y_4 >= b_{14} y_1$, $< y_2, y_2 >= b_{22} y_1$, $< y_2, y_3 >= b_{23} y_1$, $< y_2, y_4 >= b_{24} y_1$, $< y_3, y_3 >= b_{33} y_1$, $< y_3, y_4 >= b_{34} y_1$, $< y_4, y_4 >= b_{44} y_1$. By Jacobi identities, we obtain $b_{11} = 0$ and

$$b_{12} b_{22} = 0, b_{13} b_{22} = 0, b_{14} b_{22} = 0,$$

$$b_{12} b_{33} = 0, b_{13} b_{33} = 0, b_{14} b_{33} = 0,$$

$$b_{12} b_{44} = 0, b_{13} b_{44} = 0, b_{14} b_{44} = 0,$$

then we have either $b_{12} = b_{13} = b_{14} = 0$ or $b_{22} = b_{33} = b_{44} = 0$.

*Case 1.* If $b_{12} = b_{13} = b_{14} = 0$, then $< y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_1, y_3 >= 0, < y_1, y_4 >= 0$. Thus we have

$$\begin{aligned} L_4^{26} \quad : \quad & < y_1, y_1 >= 0, < y_1, y_2 >= 0, < y_1, y_3 >= 0, < y_1, y_4 >= 0, \\ & < y_2, y_2 >= b_{22} y_1, < y_2, y_3 >= b_{23} y_1, < y_2, y_4 >= b_{24} y_1, \\ & < y_3, y_3 >= b_{33} y_1, < y_3, y_4 >= b_{34} y_1, < y_4, y_4 >= b_{44} y_1; \\ & b_{22}^2 + b_{23}^2 + b_{24}^2 + b_{33}^2 + b_{34}^2 + b_{44}^2 \neq 0. \end{aligned}$$

*Case 2.* If $b_{22} = b_{33} = b_{44} = 0$, then $< y_1, y_1 >= 0, < y_2, y_2 >= 0,$ $< y_3, y_3 >= 0, < y_4, y_4 >= 0$. Thus we have

$$\begin{aligned} L_4^{27} \ : \ & < y_1, y_1 >= 0 < y_1, y_2 >= b_{12}y_1, < y_1, y_3 >= b_{13}y_1, \\ & < y_1, y_4 >= b_{14}y_1, < y_2, y_2 >= 0, < y_2, y_3 >= b_{23}y_1, \\ & < y_2, y_4 >= b_{24}y_1, < y_3, y_3 >= 0, < y_3, y_4 >= b_{34}y_1, \\ & < y_4, y_4 >= 0; b_{12}^2 + b_{13}^2 + b_{14}^2 + b_{23}^2 + b_{24}^2 + b_{34}^2 \neq 0. \end{aligned}$$

*C.* $\dim < L, L >= 2, < L, L >= \mathbf{F}y_1 + \mathrm{F}y_2$. Then we have

$$\begin{aligned} L_4^{28} \ : \ & < y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2, \\ & < y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2, < y_1, y_4 >= a_{14}^1 y_1 + a_{14}^2 y_2, \\ & < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2, < y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2, \\ & < y_2, y_4 >= a_{24}^1 y_1 + a_{24}^2 y_2, < y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2, \\ & < y_3, y_4 >= a_{34}^1 y_1 + a_{34}^2 y_2, < y_4, y_4 >= a_{44}^1 y_1 + a_{44}^2 y_2, \\ & \mathrm{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 & a_{22}^1 & a_{23}^1 & a_{24}^1 & a_{33}^1 & a_{34}^1 & a_{44}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{14}^2 & a_{22}^2 & a_{23}^2 & a_{24}^2 & a_{33}^2 & a_{34}^2 & a_{44}^2 \end{pmatrix} = 2. \end{aligned}$$

*D.* $\dim < L, L >= 3, < L, L >= \mathrm{F}y_1 + \mathrm{F}y_2 + \mathrm{F}y_3$, thus we obtain the type

$$\begin{aligned} L_4^{29} \ : \ & < y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2 + a_{11}^3 y_3, < y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2 + a_{12}^3 y_3, \\ & < y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2 + a_{13}^3 y_3, < y_1, y_4 >= a_{14}^1 y_1 + a_{14}^2 y_2 + a_{14}^3 y_3, \\ & < y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2 + a_{22}^3 y_3, < y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2 + a_{23}^3 y_3, \\ & < y_2, y_4 >= a_{24}^1 y_1 + a_{24}^2 y_2 + a_{24}^3 y_3, < y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2 + a_{33}^3 y_3, \\ & < y_3, y_4 >= a_{34}^1 y_1 + a_{34}^2 y_2 + a_{34}^3 y_3, < y_4, y_4 >= a_{44}^1 y_1 + a_{44}^2 y_2 + a_{44}^3 y_3, \\ & \mathrm{rank} \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 & a_{22}^1 & a_{23}^1 & a_{24}^1 & a_{33}^1 & a_{34}^1 & a_{44}^1 \\ a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{14}^2 & a_{22}^2 & a_{23}^2 & a_{24}^2 & a_{33}^2 & a_{34}^2 & a_{44}^2 \\ a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{14}^3 & a_{22}^3 & a_{23}^3 & a_{24}^3 & a_{33}^3 & a_{34}^3 & a_{44}^3 \end{pmatrix} = 3. \end{aligned}$$

*E.* $\dim < L, L >= 4, < L, L >= \mathrm{F}y_1 + \mathrm{F}y_2 + \mathrm{F}y_3 + \mathrm{F}y_4$, then we have

$$L_4^{30} \ : \ < y_1, y_1 >= a_{11}^1 y_1 + a_{11}^2 y_2 + a_{11}^3 y_3 + a_{11}^4 y_4,$$

$$
\begin{aligned}
&< y_1, y_2 >= a_{12}^1 y_1 + a_{12}^2 y_2 + a_{12}^3 y_3 + a_{12}^4 y_4,\\
&< y_1, y_3 >= a_{13}^1 y_1 + a_{13}^2 y_2 + a_{13}^3 y_3 + a_{13}^4 y_4,\\
&< y_1, y_4 >= a_{14}^1 y_1 + a_{14}^2 y_2 + a_{14}^3 y_3 + a_{14}^4 y_4,\\
&< y_2, y_2 >= a_{22}^1 y_1 + a_{22}^2 y_2 + a_{22}^3 y_3 + a_{22}^4 y_4,\\
&< y_2, y_3 >= a_{23}^1 y_1 + a_{23}^2 y_2 + a_{23}^3 y_3 + a_{23}^4 y_4,\\
&< y_2, y_4 >= a_{24}^1 y_1 + a_{24}^2 y_2 + a_{24}^3 y_3 + a_{24}^4 y_4,\\
&< y_3, y_3 >= a_{33}^1 y_1 + a_{33}^2 y_2 + a_{33}^3 y_3 + a_{33}^4 y_4,\\
&< y_3, y_4 >= a_{34}^1 y_1 + a_{34}^2 y_2 + a_{34}^3 y_3 + a_{34}^4 y_4,\\
&< y_4, y_4 >= a_{44}^1 y_1 + a_{44}^2 y_2 + a_{44}^3 y_3 + a_{44}^4 y_4,\\
&\operatorname{rank}\begin{pmatrix}
a_{11}^1 & a_{12}^1 & a_{13}^1 & a_{14}^1 & a_{22}^1 & a_{23}^1 & a_{24}^1 & a_{33}^1 & a_{34}^1 & a_{44}^1\\
a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{14}^2 & a_{22}^2 & a_{23}^2 & a_{24}^2 & a_{33}^2 & a_{34}^2 & a_{44}^2\\
a_{11}^3 & a_{12}^3 & a_{13}^3 & a_{14}^3 & a_{22}^3 & a_{23}^3 & a_{24}^3 & a_{33}^3 & a_{34}^3 & a_{44}^3\\
a_{11}^4 & a_{12}^4 & a_{13}^4 & a_{14}^4 & a_{22}^4 & a_{23}^4 & a_{24}^4 & a_{33}^4 & a_{34}^4 & a_{44}^4
\end{pmatrix} = 4.
\end{aligned}
$$

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ALGEBRAS GROUPS AND GEOMETRIES 28, 453- 458 (2011) 

# A NEW METHOD OF GENERATING KEY MATRIX AND ITS INVERSE FOR HILL CIPHER ALGORITHM

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## Abstract

Hill ciphers are an application of linear algebra to cryptology, the science that coding and decoding the text messages. The cryptographic involves the use of nonsingular $n \times n$ matrices with integer elements that said *key matrices* and their inverses. Key matrices must have some properties. In this paper we want to introduce a method of generating these invertible matrices in simple way.

**Key words:** Hill Cipher; Generating Key Matrix; Cryptographic; Coding; Decoding.

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# 1 Introduction

Hill Cipher is a method for transforming a text message in to a new form that is unintelligible to anyone who does not know the rule of this transformation.In fact a Hill **k-cipher** works as follows:
Assign to $k-character$ a unique integer$\{0,1,2,...,k-1\}$ respectively. These $k$ characters can be alphabet or any other sign that exist in that language such as $.,?,\sqcup$( blank space) and etc.

# 2 Main result

For instance let $\alpha_1,\alpha_2,...,\alpha_k$ be all of characters that exist, we assign to these characters $\{0,1,2,...,k-1\}$ respectively,and let

$$P=\alpha_1\alpha_2...\alpha_l$$

be a $l-character$ text where $1\leq l\leq k$.
At the first this string converted to the corresponding numerical values as bellow:

$$P=0\ 1\ 2\ ...\ l$$

Then we partition $P$ into column vectors $P_1,P_2,...,P_m$ each of them with $r$ component ($r$ is an arbitrary integer positive number) as follow:

$P_1=[0,1,...,r-1]^t,\quad P_2=[r,r+1,...2r-1]^t,...,\quad p_m=[(m-1)r,(m-1)r+1,...,l]^t$.
**Note:** if $P_m$ is not contain $r$ component we repeat the final number to fill out it.
The next step we must find a nonsingular matrix $A=[a_{ij}]_{r\times r}$ that called *Key Matrix* and multiple it from left to $P$.

$$AP=[AP_1,AP_2,...,AP_m]\qquad (mod\ k)$$

$AP$ is called a *Hill r-cipher* matrix.In other word the $P$ is coded by Hill Cipher algorithm. In order to decoding the $AP$ we need the inverse of key matrix $A$ i.e. $(A^{-1})$.
We know that $A^{-1}=\frac{1}{|A|}adj(A)$ and since the entries of $A^{-1}$ must be integer and belong to$\{0,1,2,...,k-1\}$ on the other hand since the entries of $A$ are integer and positive, so it is sufficient that we convert$\frac{1}{|A|}$ in modula of $k$, at the first we notice

to a definition and then a theorem as follow:

**Definition 2.1** Let $a$ be an integer and nonzero number, the smallest positive, integer number $x$ that is satisfy in $ax \equiv 1(mod\ k)$ called the inverse of $a$ in modula of $k$.

For example if we have $13x \equiv 1(mod\ 32)$ then $x = 5$ so $(13)^{-1} = 5$ in modula 32.

**Theorem 2.2.** Let $a, b, c \in \mathbb{Z}$ and $a$ and $b$ simultaneously nonzero, then the equation $ax + by = c$ has a solution if and only if $d = (a, b)|c$. And if $x_0, y_0$ be one of the solution the equation then the general solution obtain from:

$$x = x_0 + \frac{b}{d}t$$

$$y = y_0 - \frac{a}{d}t$$

where $t \in \mathbb{Z}$.

Now if $a$ and $K$ be given integer numbers there fore $x$ is inverse of $a$ in modula of $K$ if $ax \equiv 1(mod\ k)$. and $ax - 1 = ky$ for some $y \in \mathbb{Z}$ or $ax - ky = 1$.

From the theorem 2.2 this equation has a solution if and only if $(a, k) = 1$, so it is sufficient that the key matrix $A = [a_{ij}]_{r\times r}$ be a matrix with $|A| = a$ and $(|A|, k) = 1$. This is trivial that such a matrix is not unique, if we choose a diagonal or up triangular and or low triangular matrix that the multiple of diagonal elements be $a$ then $|A| = a$. We must solve the equation $|A|x - ky = 1$ or $x = \frac{ky+1}{|A|}$ for some $y \in \mathbb{Z}$ and find the smallest positive integer for it. And from there we have

$$\frac{1}{|A|} = x(mod\ k)$$

Finally for decoding the$AP$ we can set

$$\begin{aligned} A^{-1}(AP) &= \frac{1}{|A|}adj(A)(AP) \\ &= x(adj(A)(AP)(mod\ k) \\ &= P. \end{aligned}$$

**Example 2.3.**

Suppose that we want to code and then decode the following text with a Hill $3-$ *cipher.*

$$P = That's\ life;\ feel\ it,\ live\ it\ \&\ enjoy\ it.$$

$$\begin{pmatrix} a & b & c & d & e & f & g & h & i & j & k & l & m & n & o & p & q \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{pmatrix}$$

$$\begin{pmatrix} r & s & t & u & v & w & x & y & z & \& & . & \sqcup & ; & , & ' \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 \end{pmatrix}$$

*Table* 1 : *Numerical coding of* $32-$*letter alphabet*

With the above notations in this example $k = 32$ , $l = 41$.

$$\begin{pmatrix} t & h & a & t & ' & s & \sqcup & l & i & f & e & ; & \sqcup & f & e & e & l & \sqcup & i & t & , \\ 19 & 7 & 0 & 19 & 31 & 18 & 28 & 11 & 8 & 5 & 4 & 29 & 28 & 5 & 4 & 4 & 11 & 28 & 8 & 19 & 30 \end{pmatrix}$$

$$\begin{pmatrix} \sqcup & l & i & v & e & \sqcup & i & t & \sqcup & \& & \sqcup & e & n & j & o & y & \sqcup & i & t & . \\ 28 & 11 & 8 & 21 & 4 & 28 & 8 & 19 & 28 & 26 & 28 & 4 & 13 & 9 & 14 & 24 & 28 & 8 & 19 & 27 \end{pmatrix}$$

*Table* 2 : *Numerical coding of* $41-$*letter of the text*

Now set $r = 3$ so we have :

$$P = \begin{bmatrix} 19 & 19 & 28 & 5 & 28 & 4 & 8 & 28 & 21 & 8 & 26 & 13 & 24 & 19 \\ 7 & 31 & 11 & 4 & 5 & 11 & 19 & 11 & 4 & 19 & 28 & 9 & 28 & 27 \\ 0 & 18 & 8 & 29 & 4 & 28 & 30 & 8 & 28 & 28 & 4 & 14 & 8 & 27 \end{bmatrix}$$

We must pick an arbitrary nonsingular matrix $A = [a_{i,j}]$ that $(|A|, 32) = 1$ for example we set $|A| = 15$ and set:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Note:**We can pick every another $A$ with $|A| = 15$. For coding the $P$ we have:

$$AP = \begin{bmatrix} 25 & 25 & 20 & 15 & 20 & 12 & 24 & 20 & 31 & 24 & 14 & 7 & 8 & 25 \\ 3 & 27 & 23 & 20 & 25 & 23 & 31 & 23 & 20 & 31 & 12 & 13 & 12 & 7 \\ 0 & 18 & 8 & 29 & 4 & 28 & 30 & 8 & 28 & 28 & 4 & 14 & 8 & 27 \end{bmatrix}$$

The elements of AP are in modula 32 so the coded text is:

$$zdaz.suxipu; uzemx \sqcup y\,', uxi\,'u \sqcup y\,' \sqcup omehnoimizh.$$

Now for decoding the recent text we have to at the first to compute $A^{-1}$ as follow:

$$adj(A) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$|A| = 15 \Longrightarrow 15x - 32y = 1 \Longrightarrow x = 15 \Longrightarrow \frac{1}{|A|} = 15(mod\ 32)$

$$A^{-1} = \frac{1}{|A|}adj(A) = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}(AP) = \begin{bmatrix} 275 & 275 & 220 & 165 & 220 & 132 & 264 & 220 & 341 & 264 & 154 & 77 & 88 & 275 \\ 39 & 351 & 299 & 260 & 325 & 299 & 403 & 299 & 260 & 403 & 156 & 169 & 156 & 91 \\ 0 & 18 & 8 & 29 & 4 & 28 & 30 & 8 & 28 & 28 & 4 & 14 & 8 & 27 \end{bmatrix}$$

This matrix in modula 32 is as follow:

$$\begin{bmatrix} 19 & 19 & 28 & 5 & 28 & 4 & 8 & 28 & 21 & 8 & 26 & 13 & 24 & 19 \\ 7 & 31 & 11 & 4 & 5 & 11 & 19 & 11 & 4 & 19 & 28 & 9 & 28 & 27 \\ 0 & 18 & 8 & 29 & 4 & 28 & 30 & 8 & 28 & 28 & 4 & 14 & 8 & 27 \end{bmatrix} = P.$$

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# ON Z-TRANSFORM FOR DISTRIBUTION SPACES

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## Abstract

Analysis of distribution spaces $\mathcal{D}'$, the dual of the space $\mathcal{D}$, is introduced for the Z - transform and the basic theorems and properties are proved for this space.

**Keywords :** Z - transform, Fourier transform, testing function space, distribution spaces

**2010 Mathematics Subject Classification :** 44A10, 46F05, 46F10.

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## 1. Introduction

The integral transforms for distribution spaces are employed by many researchers as [1, 5]. In this section we introduce the Z - transform and their properties. Further, we intend to establish the Z - transform for the distribution spaces, generalize the classical Z - transform and suggest more applications. In continuous linear systems, the Fourier and the Laplace transforms are applied, which is similar to the application of the Z - transform in the linear time invariant discrete time systems. The Z - transform and its operational properties can be seen in [2, 3] .

The sampled function [2, pp. 446-448] is defined by

$$\begin{aligned} f^*(t) &= f(t)\sum_{n=0}^{\infty}\delta(t-nT) \\ &= \sum_{n=0}^{\infty} f(nT)\delta(t-nT) \qquad (1) \\ &= \sum_{n=-\infty}^{\infty} f(t_n)\delta(t-t_n) \quad , \end{aligned}$$

where $t_n$ ($n$ is an integer) is any particular value of $t$ and $f(t)$ is continuous function in any interval containing the point $t = t_n$. When $t_n = nT$, the series $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ is called the impulse train and $T$ is the sampling period.

The Laplace transform of the sampled function (1) [2, p. 449] is defined by

$$L\{f^*(t)\} = \overline{f}^*(s) = \sum_{n=0}^{\infty} f(nT)\exp(-nsT) \quad , \qquad (2)$$

where $s$ is the transform variable (which is a complex variable).

If we consider $z = \exp(sT)$, then

$$L\{f^*(t)\} = F(z) = \sum_{n=0}^{\infty} f(nT)\ z^{-n} \quad . \qquad (3)$$

Therefore, $F(z)$ is called the Z - transform of the sequence $\{f(nT)\}$, where $T$ is the sampling time. As $T$ has no effect on the properties of the Z - transform, we may consider $T = 1$. The Z - transform of a sequence $\{f(n)\}$, in terms of function $F(z)$ of a complex variable $z$ , is defined by

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)\ z^{-n} \tag{4}$$

The series in (3), converges for complex variable $z$, and the radius of convergence for the series is taken to be $R$, where $|z| > R$. Since

$$|z| = |\exp(sT)| = |\exp(\sigma + i\mu)T| = |\exp(\sigma T)| \quad , \sigma < 0, |z| < 1 \ ,$$

it follows that, when $\sigma < 0, |z| < 1$, the left half of the complex $s$-plane corresponds to the interior of the unit circle in the complex $z$ -plane. Similarly, the right half of the $s$ - plane corresponds to exterior ($|z| > 1$) of the unit circle in the $z$-plane, and when $\sigma = 0$ in the $s$-plane, it corresponds to the unit circle in the $z-$plane.

The inverse Z - transform is given by [2, p. 449]

$$Z^{-1}\{F(z)\} = f(n) = \frac{1}{2\pi i} \oint_C F(z)\ z^{n-1} dz \quad , \tag{5}$$

where $C$ is the simple closed contour, enclosing the origin and lying outside the circle $|z| = R$.

The bilateral Z - transform is defined by

$$Z\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n)\ z^{-n} \quad , \tag{6}$$

for all complex numbers $z$ for which the series converges. Substituting $z = re^{i\theta}$ and for $r = 1$, the Z - transform (6) is written as

$$F(e^{i\theta}) = \sum_{n=-\infty}^{\infty} f(n)\ e^{-in\theta} \tag{7}$$

i.e.

$$F(\theta) = \sum_{n=-\infty}^{\infty} f(n)\ e^{-in\theta} = \mathcal{F}\{f(n)\} \quad , \tag{8}$$

which apparently is the Fourier transform of the sequence $\{f(n)\}_{-\infty}^{\infty}$ . In other words, the relation between the Z - transform and the Fourier transform is

$$Z\{f(n)\} = F(e^{i\theta}) = \mathcal{F}\{f(n)\} \quad ; \ z = re^{i\theta}, r = 1$$

The basic operational properties of the Z - transform are studied in details in [2]. Few of them are

(i) *Multiplication* : If $Z\{f(n)\} = F(z)$ , then

$$Z\{nf(n)\} = -z\,\frac{d}{dz}F(z) \tag{9}$$

and

$$Z\{n^k f(n)\} = (-1)^k \quad \left(z\frac{d}{dz}\right)^k F(z) \quad , k = 0, 1, 2, \ldots \tag{10}$$

(ii) *Convolution* : If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$ , then the Z - transform of the convolution $f(n) * g(n)$ is given by

$$Z\{f(n) * g(n)\} = Z\{f(n\}Z\{g(n)\} \quad . \tag{11}$$

(iii) *Parseval's Formula* : If $F(z) = Z\{f(n)\}$ and $G(z) = Z\{g(n)\}$, then

$$\sum_{n=-\infty}^{\infty} f(n)\overline{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\theta})\overline{G}(e^{i\theta})d\theta \tag{12}$$

In particular,

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(e^{i\theta})\right|^2 d\theta \quad . \tag{13}$$

Definitions of $L^p$spaces and $H^p$ spaces for continuous and discrete Fourier, Laplace and the Z - transforms can be seen in [4]. For discrete functions, $L^p$ spaces are denoted as $l^p$, suggested by

$$\sum_{k=-\infty}^{\infty} \|K(k)\|^p < \infty$$

For $1 \leq p \leq \infty$, the norm in $l^p$ - space is

$$\|K\|_p = \left(\sum_{k=-\infty}^{\infty} \|K(k)\|^p\right)^{1/p} \quad , \tag{14}$$

whereas the norm in $L^p$ - spaces for continuous (or continuous time) functions is given by

$$\|K\|_p = \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} \|K(t)\|^p\, dt\right)^{1/p} < \infty \quad . \tag{15}$$

On considering $f$ to be an analytic function, the Laplace and the Z - transform are defined for $H^p$ spaces also.

The Z - transform of the sequence $\{u(k)\}_{k=0}^{\infty} \in l^2(Z^+; C^n)$ is defined by

$$\widehat{u}(z) = \sum_{k=0}^{\infty} z^{-k} u(k) \quad . \tag{16}$$

Considering $z = re^{i\varphi}$ , the Z - transform is equivalent to the discrete Fourier transform of the sequence $\{r^{-k} z(k)\}_{k=0}^{\infty}$. For $r \geq 1$,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \left\| \widehat{u}(re^{i\varphi}) \right\|^2 d\varphi &= \sum_{k=0}^{\infty} r^{-2k} \left\| u(k) \right\|^2 \\ &\leq \sum_{k=0}^{\infty} \left\| u(k) \right\|^2 = \left\| u \right\|_2^2 \quad . \end{aligned} \tag{17}$$

Therefore, the Z - transform $\widehat{u}$ is bounded and analytic, and

$$\frac{1}{2\pi} \int_0^{2\pi} \left\| \widehat{u}(re^{i\varphi}) \right\|^2 d\varphi \leq M \quad . \tag{18}$$

Similarly, the mapping from one space to another, for $L^p$ and $H^p$ spaces, are given for the Z - transform.

In what follows, we introduce the brief definitions and notations of the distribution spaces and prove some theorems for the Z - transform on distribution spaces.

## 2. Distribution Spaces for Z - Transform

The testing function space and its dual, which is also known as distribution spaces (i.e. generalized functions) are employed. In what follows in the remaining text of the paper are definitions and properties of the distribution spaces, see [5].

The space of testing functions, which is denoted by $\mathcal{D}$, consists of all complex valued functions $\varphi(t)$ that are infinitely smooth and vanishes outside some finite interval.

A sequence of testing functions $\{\varphi_v(t)\}_{v=1}^{\infty}$ is said to converge in $\mathcal{D}$ if $\varphi_v(t)$ are in $\mathcal{D}$, if they are all zero outside some fixed finite interval $I$, and if for every fixed non - negative integer $k$ the sequence $\{\varphi_v^{(k)}(t)\}_{v=1}^{\infty}$ converges uniformly for $-\infty < t < \infty$.

The functionals that assign a complex number to every member of $\mathcal{D}$ is denoted as $\langle f, \varphi \rangle$ which have two properties, linearity and continuity.

*Linearity* : The functional $f$ on $\mathcal{D}$ is said to be linear if testing functions $\varphi_1$ and $\varphi_2$ and any complex number $\alpha$ form relations

$$\left.\begin{array}{c} \langle f, \varphi_1 + \varphi_2 \rangle = \langle f, \varphi_1 \rangle + \langle f, \varphi_2 \rangle \\ \langle f, \alpha\varphi_1 \rangle = \alpha \langle f, \varphi_1 \rangle \end{array}\right\} \quad . \tag{19}$$

*Continuity* : A functional $f$ on $\mathcal{D}$ is said to be continuous if for any sequence of testing functions $\{\varphi_\upsilon(t)\}_{\upsilon=1}^{\infty}$ that converges to $\varphi(t)$ in $\mathcal{D}$.

In other words, a continuous linear functional on the space $\mathcal{D}$ is a distribution. The space of all such distributions is denoted by $\mathcal{D}'$ ($\mathcal{D}'$ is called the dual (or conjugate) space of $\mathcal{D}$).

### 2.1 Distributional Z - Transform

The locally integrable function (if a function is absolutely continuous or integrable on each closed and bounded subinterval of the interval $I$, it is called locally absolutely continuous or locally integrable) $f(t)$ is absolutely integrable over $-\infty < t < \infty$

$$\int_{-\infty}^{\infty} |f(t)| < \infty \quad .$$

The Z - transform $F(z)$ of $f(n)$ is a function of $z$ and is defined by

$$F(z) = Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n)\ z^{-n} \ .$$

$F(z)$ is bounded for all $z$ if we write

$$\begin{aligned} |F(z+y) - F(z)| &= \sum_{n=-\infty}^{\infty} f(n)\ (z+y)^{-n} - \sum_{n=-\infty}^{\infty} f(n)\ z^{-n} \\ &\leq \sum_{n=-\infty}^{\infty} |f(n)|\ \left|(z+y)^{-n} - z^{-n}\right| \\ &\leq \varepsilon \end{aligned} \tag{20}$$

Moreover, if we consider the Z - transform, as defined in (17) and (18), then (for apparent reasons) it is justified that the Z - transform is bounded and analytic for $l^p-$ space and $H^p$ - space, respectively. We can write

$$F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} \langle f(n), z^{-n} \rangle \quad , \tag{21}$$

where $f(n) \in \mathcal{D}'$ and $z \in \mathcal{D}$. This is said to be the distributional Z - transform.

**Theorem 1 :** *If the locally integrable functions $f(n)$ and $g(n)$ are absolutely integrable over $n$, then*

$$\sum_{n=-\infty}^{\infty} f(n)\overline{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta})\overline{G}(e^{i\theta}) d\theta \ . \tag{22}$$

*In particular,*

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|F(e^{i\theta})\right|^2 d\theta \quad . \tag{23}$$

**Proof :** If $F(z)$ and $G(z)$ are bounded and continuous for all $z$, then (22) converges. Moreover,

$$\sum_{n=-\infty}^{\infty} f(n)\overline{g}(n) = \frac{1}{2\pi i} \int_{C} F(z) z^{n-1} \sum_{n=-\infty}^{\infty} \overline{g}(n) \quad .$$

If we consider $z = re^{i\theta}$ and $r = 1$, then for $dz = e^{i\theta} i d\theta$, the right hand side will be

$$\begin{aligned} &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \overline{g}(n) \int_{C} F(e^{i\theta})(e^{i\theta})^{n-1} i e^{i\theta} d\theta \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \overline{g}(n) e^{i\theta n} \int_{-\pi}^{\pi} F(e^{i\theta}) d\theta \quad . \end{aligned}$$

From (17) and (18), we express $F(e^{i\theta})$ is an absolutely integrable function,

$$\sum_{n=-\infty}^{\infty} f(n)\overline{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta})\overline{G}(e^{i\theta}) d\theta \quad .$$

Since $F(e^{i\theta})$ and $G(e^{i\theta})$ are locally and absolutely integrable functions and $f(n)$ and $g(n)$ belong to $l^p$(or $l^2$) spaces, in particular, we can write (22) in the form

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|F(e^{i\theta})\right|^2 d\theta \quad ,$$

which proves (23). In other words, we can say that the Parseval formula (or Plancherel formula) of the Z - transform, as stated in (12) and (13), is also proved for distribution spaces $\mathcal{D}'$.

**Theorem 2 :** *If $f$ and $g$ are Z - transformable distributions in $D'$ and $F(z) = Z\{f(n)\}; G(z) = Z\{g(n)\}$, then $f * g$ is a Z - transformable distribution in $D'$.*

$$Z\{f * g\} = F(z)G(z) \quad . \tag{24}$$

**Proof :** Since the sequence $f(n)$ and $g(n)$ are bounded and analytic,

$$\begin{aligned} Z\{f(n) * g(n)\} &= \sum_{n=0}^{\infty} z^{-n} \sum_{m=0}^{\infty} f(n-m)g(m) \quad , \\ &= \sum_{m=0}^{\infty} g(m) \sum_{n=0}^{\infty} f(n-m)z^{-n} \quad . \end{aligned}$$

On considering $(n - m) = r$, we have

$$Z\{f(n) * g(n)\} = \sum_{m=0}^{\infty} g(m)z^{-m} \sum_{r=-m}^{\infty} f(r)z^{-r}$$

As $f(r) = 0, r < 0$, we write

$$Z\{f(n) * g(n)\} = \sum_{m=0}^{\infty} g(m)z^{-m} \sum_{r=-m}^{\infty} f(r)z^{-r}$$

i.e.

$$= Z\{f(n)\}Z\{g(n)\} = F(z)G(z) \quad .$$

Assuming $f(n) = 0 = g(n)$ for $n < 0$, the convolution of the Z - transform assumes the form

$$Z\{f(n) * g(n)\} = \sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} f(n-m)g(m) \quad .$$

Changing the order of summation, we further have

$$\begin{aligned} Z\{f(n) * g(n)\} &= \sum_{m=-\infty}^{\infty} g(m) \sum_{m=0}^{\infty} f(n-m)z^{-n} \quad , \\ &= \sum_{m=-\infty}^{\infty} g(m)z^{-m} \sum_{n=-\infty}^{\infty} f(n-m)z^{-(n-m)} \end{aligned}$$

i.e.

$$= \sum_{m=-\infty}^{\infty} g(m)z^{-m} \sum_{r=-\infty}^{\infty} f(r)z^{-r} \quad ; r = n - m$$

Thus,

$$Z\{f(n) * g(n)\} = Z\{f(n)\}Z\{g(n)\} \quad .$$

Owing to the relation connecting the Z - transform and the Fourier transform [see equation (8) and the following relation], Theorem 2 (the convolution) supports bilateral Z - transform (6). Further, this theorem establishes the fact that the Z - transform is, indeed, transformable in $\mathcal{D}'$.

As $Z\{f(n)\} = F(z)$ is on distribution spaces which is applicable when $f(n) \in \mathcal{D}'$, then the basic operational formula as defined in Section 1 and in [2, 3] such as multiplication, differentiation, and others will be applicable in the space $\mathcal{D}'$.

**Concluding Remark:** Since $f(n) \in \mathcal{D}', Z\{f(n)\} = F(z)$, the operation formulae discussed in Section 1 and those in [2, 3] are in $\mathcal{D}'$. It may not be out of place to mention that the properties of distributional Z - transform can assist the study of the Z - transform in distribution spaces $\mathcal{E}'$ and $S'$ (we believe such investigations are yet to be ventured).

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